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Almost split real forms for hyperbolic Kac–Moody Lie algebras

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Abstract

A Borel–Tits theory was developed for almost split forms of symmetrizable Kac–Moody Lie algebras. In this paper, we look to almost split real forms and their restricted root systems for symmetrizable hyperbolic Kac–Moody Lie algebras. We establish a complete list of these forms, in terms of their Satake–Tits index, for the strictly hyperbolic ones and for those which are obtained as (hyperbolic) canonical Lorentzian extensions of affine Lie algebras. These forms are of particular interest in theoretical physics because of their connection to supergravity theories.

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1. Introduction

Since their appearance in the late 1960s, as generalizations of semi-simple complex Lie algebras, the (infinite-dimensional) Kac–Moody Lie algebras have played an increasingly crucial role in various areas of mathematics as well as theoretical physics. The hyperbolic Kac–Moody Lie algebras (which constitute a subclass of Lorentzian Kac–Moody algebras [24]) and some of their (almost split) real forms have appeared, besides the affine Kac–Moody algebras, in a variety of problems in the realms of string theory ([14, 13], . . .) and supergravity theories ([36, 16], . . .).

Almost split forms of symmetrizable Kac–Moody Lie algebras were studied in [30–32] and [2] for an arbitrary field of characteristic 0: a Borel–Tits Theory was developed for these forms and a classification, in the real case, in terms of the Satake–Tits index, was done for affine Lie algebras [2]. The relative (or restricted) root systems were also considered in [2]. In [29], G Rousseau gave a realization, in terms of the loop algebras, for all the almost split real forms of affine Lie algebras. The same construction was done by V Back for an arbitrary field of characteristic 0 instead of the real field ([2], section 5). Some

forms (which may be almost anisotropic or almost compact in the real case) of symmetrizable Kac–Moody algebras are defined by generators and relations [1, 10]. Almost compact real forms of affine Kac–Moody algebras were studied in [4] and [28] and entirely classified in [7]. The conjugate classes of their Cartan subalgebras were classified in [8].

This paper is devoted to the classification (in terms of the Satake–Tits index) of almost split real forms, and their relative root systems, for some symmetrizable hyperbolic Kac–Moody Lie algebras (namely, the strictly hyperbolic Kac–Moody algebras and Kac–Moody Lie algebras which are obtained as (canonical) Lorentzian extensions of affine Lie algebras) which we consider the most met in supergravity theories [16].

The paper is organized as follows. In section 2, we recall the construction of (symmetrizable) Kac–Moody Lie algebras and groups from the so-called generalized Cartan matrices and we set the notation. We also give a description of the automorphisms group and the invariant bilinear form for any indecomposable and symmetrizable Kac–Moody algebra when the defining generalized Cartan matrix is non-singular (since in this case description is less complicated, and it is in particular valid for the hyperbolic case).

Sections 3 and 4 are mostly an exposition of known results on almost split forms [2, 3] that we have written here, in the real case, for hyperbolic Kac–Moody algebras. The main results are theorems 3.2.7, 3.3.1 and 3.6.3 which are also valid for any symmetrizable Kac–Moody algebra with non-singular generalized Cartan matrix. In particular, we prove in section 3 (theorem 4.1.1) that the Iwasawa decomposition holds for these forms and the corresponding real Kac–Moody groups.

In section 5, we introduce, in a more general way, the notion of Lorentzian extensions of affine Lie algebras and we enumerate those which are hyperbolic. Finally, we apply the results of sections 3 and 4 to give a complete list of almost split real forms, together with their restricted root systems, for the subclass of strictly hyperbolic Kac–Moody Lie algebras and hyperbolic Kac–Moody Lie algebras which are obtained as (canonical) Lorentzian extensions of affine Lie algebras. Note that the classification of almost split real forms could be done for any symmetrizable hyperbolic Kac–Moody Lie algebra, but as the list of these algebras is long (at least 136) we have restricted ourselves to the selected subclass of hyperbolic algebras cited above.

2. Preliminaries

We recall the most important known facts about Kac–Moody algebras and groups, we add some facts in the hyperbolic case and set the notation.

2.1. Generalized Cartan matrices

An $n \times n$ matrix $A = (a_{i,j})$ is called a *generalized Cartan matrix* if it satisfies

- (1) $a_{i,i} = 2$ ($i = 1, 2, \dots, n$),
- (2) $a_{i,j} \in \{0, -1, -2, \dots\}$ ($i \neq j$),
- (3) $a_{i,j} = 0$ implies $a_{j,i} = 0$.

The matrix A is called *decomposable* if for a suitable permutation π of $\{1, 2, \dots, n\}$ it takes the form $(a_{\pi(i),\pi(j)}) = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where B and C are square matrices. If A is not decomposable, it is called *indecomposable*.

The matrix A is called *symmetrizable* if there exists an invertible diagonal matrix $D = \text{diag}(d_1, d_2, \dots, d_n)$ such that DA is symmetric. The entries d_1, d_2, \dots, d_n can be

chosen to be positive rational and if moreover the matrix A is indecomposable, then these entries are unique up to a constant factor.

It was stated in [17] that the collection of indecomposable generalized Cartan matrices is divided into three mutually exclusive types: *finite*, *affine* and *indefinite*. A classification of these matrices in terms of their Dynkin diagrams is done for the finite and the affine type ones and it can be found (for example) in [17], chapter 4.

An indecomposable generalized Cartan matrix A is called *strictly hyperbolic* (resp. *hyperbolic*) if it is of indefinite type and all of its principal submatrices have all their indecomposable constituents of finite (resp. finite or affine) type. This means that the Dynkin diagram corresponding to A becomes a disjoint union of Dynkin diagrams of finite (resp. finite or affine) type upon deletion of any one vertex and the edges connected to it.

The matrices $H_{s,t} = \begin{pmatrix} 2 & -s \\ -t & 2 \end{pmatrix}$, where s and t are positive integers such that $st > 4$, cover all the rank-2 generalized Cartan matrices of indefinite type, they are symmetrizable and strictly hyperbolic and they have been treated by Lepowsky and Moody in [22]. The Dynkin diagram corresponding to the strictly hyperbolic matrix $H_{s,t}$ (we may assume $0 < s \leq t$) is the following:

$$H_{(s,t)} : \begin{array}{c} \bullet \text{---} \bullet \\ \underset{1}{\bullet} \quad \quad \quad \underset{2}{\bullet} \end{array}$$

The classification by Saçlıoğlu [33] of hyperbolic Dynkin diagrams shows that there is no strictly hyperbolic (resp. hyperbolic) generalized Cartan matrix of order larger than 4 (resp. 10). The number of hyperbolic generalized Cartan matrices of order from 3 to 10 is necessarily finite (cf [33]). Note that a symmetrizable hyperbolic generalized Cartan matrix is non-singular and the corresponding symmetric matrix has signature $(+ + \cdots + -)$ cf [25]. In particular, hyperbolic generalized Cartan matrices are Lorentzian.

2.2. Kac–Moody algebras and groups

(See [17] and [27].) Let $A = (a_{i,j})$ be an $n \times n$ indecomposable and symmetrizable generalized Cartan matrix with a realization $(\mathfrak{h}, \Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}, \Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\})$, where \mathfrak{h} is a vector space over the complex field \mathbb{C} such that $\dim \mathfrak{h} = n + \text{corank}(A)$, Π and Π^\vee are linearly independent in \mathfrak{h}^* and \mathfrak{h} , respectively, such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. It follows that if A is non-singular then Π^\vee (resp. Π) is a basis of \mathfrak{h} (resp. \mathfrak{h}^*) and we denote by $P = (p_1, p_2, \dots, p_n)$ the dual basis of Π in \mathfrak{h} .

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the complex Kac–Moody algebra associated with A : it is generated by $\{\mathfrak{h}, e_i, f_i; i = 1, 2, \dots, n\}$ with the following relations:

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &= 0, & [e_i, f_j] &= \delta_{i,j} \alpha_i^\vee & (i, j = 1, 2, \dots, n); \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i, & [h, f_i] &= -\langle \alpha_i, h \rangle f_i & (h \in \mathfrak{h}); \\ (\text{ad } e_i)^{1-a_{i,j}}(e_j) &= 0, & (\text{ad } f_i)^{1-a_{i,j}}(f_j) &= 0 & (i \neq j). \end{aligned} \tag{2.2.1}$$

The Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is called of finite, affine or indefinite type if the corresponding generalized Cartan matrix A is.

The derived algebra \mathfrak{g}' of \mathfrak{g} is generated by the *Chevalley generators* e_i, f_i ($i = 1, 2, \dots, n$); and the centre \mathfrak{c} of \mathfrak{g} lies in $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \sum_{i=1}^n \mathbb{C} \alpha_i^\vee$. If the generalized Cartan matrix A is non-singular (that is the case when A is of finite type or hyperbolic) then $\mathfrak{g} = \mathfrak{g}'$ is a (finite or infinite)-dimensional simple Lie algebra and the centre \mathfrak{c} is trivial.

The *Chevalley Cartan involution* ω of \mathfrak{g} is the involutive automorphism such that $\omega(h) = -h$ ($h \in \mathfrak{h}$) and $\omega(e_i) = -f_i$ ($i = 1, 2, \dots, n$).

The subalgebra \mathfrak{h} is a maximal $\text{ad}(\mathfrak{g})$ -diagonalizable subalgebra of \mathfrak{g} , it is called the *standard Cartan subalgebra* of \mathfrak{g} . Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system; then Π is a root basis of Δ and $\Delta = \Delta^+ \cup \Delta^-$, where $\Delta^\pm = \Delta \cap \mathbb{Z}^\pm \Pi$ is the set of positive (or negative) roots relative to the basis Π .

The *Weyl group* W of $(\mathfrak{g}, \mathfrak{h})$ is generated by the fundamental reflections r_i ($i = 1, 2, \dots, n$) such that $r_i(h) = h - \langle \alpha_i, h \rangle \check{\alpha}_i$ for $h \in \mathfrak{h}$, it is a Coxeter group on $\{r_1, r_2, \dots, r_n\}$ with length function $w \mapsto l(w)$, $w \in W$. The Weyl group W acts on \mathfrak{h}^* and Δ , we set $\Delta^{\text{re}} = W(\Pi)$ (the real roots) and $\Delta^{\text{im}} = \Delta \setminus \Delta^{\text{re}}$ (the imaginary roots). Any root basis of Δ is W -conjugate to Π or $-\Pi$. The opposite root basis $-\Pi$ is W -conjugate to Π if and only if the generalized Cartan matrix A is of finite type.

For $\alpha \in \Delta$, let \mathfrak{g}_α be the root space of \mathfrak{g} corresponding to the root α . We have the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$.

A *Borel subalgebra* of \mathfrak{g} is a maximal completely solvable subalgebra. A *parabolic subalgebra* of \mathfrak{g} is a (proper) subalgebra containing a Borel subalgebra. The *standard positive (or negative) Borel subalgebra* is $\mathfrak{b}^\pm := \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha)$. A parabolic subalgebra \mathfrak{p}^+ (resp. \mathfrak{p}^-) containing \mathfrak{b}^+ (resp. \mathfrak{b}^-) is called *positive (resp. negative) standard parabolic subalgebra* of \mathfrak{g} ; then there exists a subset J of Π (different from Π) such that $\mathfrak{p}^\pm = \mathfrak{p}^\pm(J) := (\bigoplus_{\alpha \in \Delta_J} \mathfrak{g}_\alpha) + \mathfrak{b}^\pm$, where $\Delta_J = \Delta \cap (\mathbb{Z}J)$, and we say that \mathfrak{p}^\pm is of type J (cf [20]). If J is of finite type (i.e., Δ_J is finite), the standard parabolic subalgebra \mathfrak{p}^\pm is called of finite type.

In [27], Peterson and Kac construct a group G , which is the connected and simply connected complex algebraic group associated with \mathfrak{g} when \mathfrak{g} is of finite type, depending only on the derived Lie algebra \mathfrak{g}' and acting on \mathfrak{g} via the adjoint representation $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$. It is generated by the one-parameter subgroups $U_\alpha = \exp(\mathfrak{g}_\alpha)$, $\alpha \in \Delta^{\text{re}}$, and $\text{Ad}(U_\alpha) = \exp(\text{ad}_{\mathfrak{g}_\alpha})$.

For each $i = 1, 2, \dots, n$ there exists a monomorphism $\phi_i : SL_2(\mathbb{C}) \rightarrow G$ satisfying

$$\phi_i \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(te_i), \quad \phi_i \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp(tf_i), \quad t \in \mathbb{C}.$$

Set $m_i = \phi_i \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \exp(e_i) \exp(-f_i) \exp(e_i) = \exp(-f_i) \exp(e_i) \exp(-f_i)$; then $\text{Ad}(m_i)$ induces the fundamental reflection r_i on \mathfrak{h} and we may always view an element of the Weyl group W as an automorphism of \mathfrak{g} .

Let N (resp. H) be the stabilizer (resp. fixator) of \mathfrak{h} in G , then H is a normal subgroup of N and N/H is isomorphic to the Weyl group W .

The centre $Z(G)$ ($= \text{Ker}(\text{Ad})$) of G is contained in H , it is isomorphic to $\text{Hom}(\mathbb{Z}^n / A\mathbb{Z}^n, \mathbb{C}^\times)$ and it is finite when the generalized Cartan matrix A is non-singular.

The Cartan subalgebras of \mathfrak{g} are G -conjugate. If \mathfrak{g} is not of finite type, there are exactly two conjugate classes (under the adjoint action of G) of Borel subalgebras: $G \cdot \mathfrak{b}^+$ and $G \cdot \mathfrak{b}^-$. A Borel subalgebra \mathfrak{b} of \mathfrak{g} which is G -conjugate to \mathfrak{b}^+ (resp. \mathfrak{b}^-) is called positive (resp. negative). It follows that any parabolic subalgebra \mathfrak{p} of \mathfrak{g} is G -conjugate to a standard positive (or negative) parabolic subalgebra and we say that \mathfrak{p} is positive (or negative).

2.3. Automorphisms of Kac–Moody algebras

We give here a description of the group $\text{Aut}(\mathfrak{g})$ of automorphisms of \mathfrak{g} when the indecomposable and symmetrizable generalized Cartan matrix A is *non-singular* and of *indefinite type* (where it is always the case when A is of hyperbolic type). More details can be found in [2] or [20] for the general case.

A linear or semi-linear automorphism σ of \mathfrak{g} over \mathbb{C} is called of the *first kind* (resp. *second kind*) if it transforms a Borel subalgebra into a Borel subalgebra of the same (resp. opposite)

sign. Let $\text{Aut}_1(\mathfrak{g})$ denote the subgroup of automorphisms of \mathfrak{g} of the first kind. If σ is of the first kind, then $\omega\sigma$ is of the second kind and one can see easily that $\text{Aut}_1(\mathfrak{g})$ and $\omega\text{Aut}_1(\mathfrak{g})$ form a partition of $\text{Aut}(\mathfrak{g})$.

The *adjoint group* $\text{Ad}(G)$ is a normal subgroup of $\text{Aut}_1(\mathfrak{g})$ and $\text{Aut}(\mathfrak{g})$. As the generalized Cartan matrix A is assumed to be non-singular, the adjoint group is exactly the inner automorphisms group (denoted as $\text{Int}(\mathfrak{g})$ in [2]) since in this case the group $\tilde{H} = \text{Hom}(\bigoplus_{i=1}^n \mathbb{Z}\alpha_i, \mathbb{C}^\times)$ acts on \mathfrak{g} as $\text{Ad}(H)$.

Let $\text{Aut}(A)$ be the group of all permutations ρ of $\{1, 2, \dots, n\}$ such that $a_{\rho(i), \rho(j)} = a_{i, j}$. We view $\text{Aut}(A)$ as a subgroup of the group $\text{Aut}(\mathfrak{g})$ by requiring $\rho(e_i) = e_{\rho(i)}$ and $\rho(f_i) = f_{\rho(i)}$. Clearly, the group $\text{Aut}(A)$ is contained in $\text{Aut}_1(\mathfrak{g})$, it commutes with the Chevalley Cartan involution ω . Let $\text{Out}(A)$ denote the group $\{1, \omega\} \times \text{Aut}(A)$, then we have

$$\text{Aut}_1(\mathfrak{g}) = \text{Aut}(A) \ltimes \text{Ad}(G) \quad \text{Aut}(\mathfrak{g}) = \text{Out}(A) \ltimes \text{Ad}(G). \tag{2.3.1}$$

2.4. The invariant bilinear form

(See [17].) We recall that the generalized Cartan matrix A is supposed indecomposable and symmetrizable. There exists a nondegenerate $\text{ad}(\mathfrak{g})$ -invariant symmetric \mathbb{C} -bilinear form (\cdot, \cdot) on \mathfrak{g} , which is entirely determined by its restriction to \mathfrak{h} , such that

$$(\alpha_i^\check, h) = \frac{(\alpha_i^\check, \alpha_i^\check)}{2} (\alpha_i, h), \quad i = 1, 2, \dots, n, \quad h \in \mathfrak{h}.$$

Set $d_i = \frac{2}{(\alpha_i^\check, \alpha_i^\check)}$ and $D = \text{diag}(d_1, d_2, \dots, d_n)$, then the matrix DA is symmetric and we may thus assume that

$$(\alpha_i^\check, \alpha_i^\check) \text{ is a positive rational for all } i. \tag{2.4.1}$$

The form (\cdot, \cdot) is clearly $\text{Ad}(G)$ -invariant and invariant by the Chevalley Cartan involution ω (since ω acts by -1 on \mathfrak{h}). If moreover the generalized Cartan matrix A is non-singular, then the invariant bilinear form (\cdot, \cdot) satisfying the condition (2.4.1) is unique up to a positive rational factor. It follows that the form (\cdot, \cdot) is $\text{Aut}(A)$ -invariant; indeed, if ρ is a diagram automorphism, then the invariant bilinear form $(\rho(x), \rho(y))$ satisfies the condition (2.4.1); hence, there exists a positive rational λ such that $(\rho(x), \rho(y)) = \lambda(x, y)$, $\forall x, y \in \mathfrak{g}$; but ρ is of finite order and we have necessarily $\lambda = 1$. Then we deduce from (2.3.1) that the bilinear form (\cdot, \cdot) is $\text{Aut}(\mathfrak{g})$ -invariant.

3. Almost split real forms

From now on we suppose that the generalized Cartan matrix A is indecomposable, symmetrizable and non-singular of indefinite type. The associated Kac–Moody Lie algebra \mathfrak{g} is defined as in (2.2).

3.1. Definitions and notation

A *real form* $\mathfrak{g}_{\mathbb{R}}$ of \mathfrak{g} corresponds to a semi-involution (or a conjugate-linear involution) σ' of \mathfrak{g} such that $\mathfrak{g}_{\mathbb{R}}$ is the fixed point real subalgebra $\mathfrak{g}^{\sigma'}$. The real form $\mathfrak{g}_{\mathbb{R}}$ is said to be *almost split* (resp. *almost compact*) if the corresponding semi-involution σ' is of the first kind (resp. second kind).

The real subalgebra of \mathfrak{g} (viewed as a real Lie algebra) generated by $e_i, f_i, i = 1, 2, \dots, n$, is an almost split real form of \mathfrak{g} ; it is called the *standard split form* and the corresponding semi-involution of the first kind σ'_n is called the *standard normal semi-involution* of \mathfrak{g} .

The *standard Cartan semi-involution* of \mathfrak{g} is $\omega' = \sigma'_n \omega = \omega \sigma'_n$, it is of the second kind and the corresponding almost compact form is the *standard compact form* of \mathfrak{g} . A *Cartan semi-involution* of \mathfrak{g} is a G -conjugate of ω' .

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of \mathfrak{g} and σ' the corresponding semi-involution of \mathfrak{g} . If V is a σ' -stable \mathbb{C} -subspace of \mathfrak{g} we denote by $V_{\mathbb{R}} := V^{\sigma'}$ the fixed point set of σ' in V . Conversely, if $V_{\mathbb{R}}$ is a \mathbb{R} -subspace of $\mathfrak{g}_{\mathbb{R}}$ we denote by V the σ' -stable \mathbb{C} -subspace $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of \mathfrak{g} . The semi-involution σ' acts on the Kac–Moody group G associated with \mathfrak{g} and we denote by $G_{\mathbb{R}} := G^{\sigma'}$ the fixed point subgroup of σ' .

A *Cartan subalgebra* of $\mathfrak{g}_{\mathbb{R}}$ is a subalgebra whose complexification is a Cartan subalgebra of \mathfrak{g} . A *split toral subalgebra* of $\mathfrak{g}_{\mathbb{R}}$ is an $\text{ad}(\mathfrak{g}_{\mathbb{R}})$ -diagonalizable subalgebra. Any split toral subalgebra $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ is contained in a Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$. If $\mathfrak{t}_{\mathbb{R}}$ is a *maximal split toral subalgebra*, then $\mathfrak{h}_{\mathbb{R}}$ is called a *maximally split* (or *maximally noncompact*) *Cartan subalgebra*.

3.2. The index of an almost split real form

A Borel–Tits theory of almost split \mathbb{K} -forms of Kac–Moody Lie algebras, where \mathbb{K} is an arbitrary field of characteristic 0, was developed by Rousseau in [30, 31] and [2]. We recall (and rewrite in the real case) the main results on these forms for the class of Kac–Moody algebras that we have fixed above.

Proposition 3.2.1. *Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of \mathfrak{g} and σ' the corresponding semi-involution. Then $\mathfrak{g}_{\mathbb{R}}$ is almost split iff σ' stabilizes a proper parabolic subalgebra of \mathfrak{g} . In particular, almost compact real forms of \mathfrak{g} are almost \mathbb{R} -anisotropic (i.e. there is no σ' -stable proper parabolic subalgebra).*

If the real form $\mathfrak{g}_{\mathbb{R}}$ is almost split then σ' stabilizes a finite-type parabolic subalgebra \mathfrak{p} of \mathfrak{g} .

Theorem 3.2.2. *Let $\mathfrak{g}_{\mathbb{R}}$ be an almost split real form of \mathfrak{g} and $\epsilon = +$ or $-$, then the group $G_{\mathbb{R}}$ is transitive on pairs $(\mathfrak{t}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}^{\epsilon})$ where $\mathfrak{t}_{\mathbb{R}}$ is a maximal split toral subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}^{\epsilon}$ is a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ of sign ϵ and containing $\mathfrak{t}_{\mathbb{R}}$.*

Definition 3.2.3. *The rank of $\mathfrak{g}_{\mathbb{R}}$ is the common \mathbb{R} -dimension r of its maximal split toral subalgebras.*

Remark 3.2.4. The centralizer $\mathfrak{l}_{\mathbb{R}}$ of $\mathfrak{t}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ is the *Levi subalgebra* of $\mathfrak{p}_{\mathbb{R}}^{\epsilon}$, its derived algebra $\mathfrak{l}'_{\mathbb{R}}$ is a compact real semi-simple Lie algebra.

The maximal split toral subalgebra $\mathfrak{t}_{\mathbb{R}}$ is contained in a unique (up to a conjugation by $Z_{G_{\mathbb{R}}}(\mathfrak{t}_{\mathbb{R}})$) maximally split Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$. In particular, the maximally split Cartan subalgebras of $\mathfrak{g}_{\mathbb{R}}$ are $G_{\mathbb{R}}$ -conjugate.

Take a G -conjugate of σ' , we may assume that \mathfrak{h} ($= \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$) is the standard Cartan subalgebra of \mathfrak{g} and $\mathfrak{p}^{\epsilon} = \mathfrak{p}^{\epsilon}(X)$ is a finite-type standard parabolic subalgebra of sign ϵ . Therefore, the two opposite parabolic subalgebras $\mathfrak{p}^{+}(X)$ and $\mathfrak{p}^{-}(X)$ are σ' -stable and we have

$$\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{b}^{+} \subset \mathfrak{p}^{+}(X). \quad (3.2.1)$$

Thus, the pair $(\mathfrak{h}, \mathfrak{b}^{+})$ (resp. $(\mathfrak{t}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}^{+}(X))$) is called a *standardization* of \mathfrak{g} (resp. $\mathfrak{g}_{\mathbb{R}}$) and we say that the standardization $(\mathfrak{h}, \mathfrak{b}^{+})$ is compatible with $(\mathfrak{t}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}^{+}(X))$.

Definition 3.2.5 (the $*$ -action of σ'). *Actually, the derived Levi subalgebra \mathfrak{l}' is that of $\mathfrak{p}^{+}(X)$, it is the subalgebra \mathfrak{g}_X generated by \mathfrak{g}_{α} , $\alpha \in \Delta_X$. Note that $\sigma'(\mathfrak{b}^{+})$, as \mathfrak{b}^{+} , is a positive Borel subalgebra satisfying the condition (3.2.1); hence there exists $w_X \in W_X$ (the Weyl group of*

$\Delta(\mathfrak{h}, \mathfrak{l})$ viewed as a subgroup of W) such that $w_X \sigma'(\mathfrak{b}^+) = \mathfrak{b}^+$ and thus $w_X \sigma'$ induces on Π an involutive (or a trivial) diagram automorphism σ'^* called the $*$ -action of σ' .

If $X = \emptyset$ (i.e. σ' stabilizes the standard Borel subalgebra \mathfrak{b}^+) then the real form $\mathfrak{g}_{\mathbb{R}}$ is called split or quasi-split according to whether the $*$ -action σ'^* is trivial or not.

One knows ([2]; proposition 2.7) that \mathfrak{t} is the subspace of \mathfrak{h} defined by the following equations:

$$\alpha(h) = 0 \quad \forall \alpha \in X \quad \text{and} \quad \alpha(h) = \beta(h) \quad \text{if } \alpha, \beta \in \Pi \text{ satisfy } \beta = \sigma'^*(\alpha) \quad (3.2.2)$$

The restriction of σ' to the derived Levi subalgebra \mathfrak{l}' is compact and we may assume that σ' coincides with the standard Cartan semi-involution ω' on \mathfrak{l}' . In particular, σ' commutes with w_X on \mathfrak{h} and so $w_X^2 = 1_W$ is the unity of W .

Definition 3.2.6 (the index of $\mathfrak{g}_{\mathbb{R}}$). *The Satake–Tits index of $\mathfrak{g}_{\mathbb{R}}$ is the data consisting of the Dynkin diagram of \mathfrak{g} , the $*$ -action of σ' on it and $X = \text{type}(\mathfrak{p}^+(X))$.*

Schematically, the Satake–Tits index is the so-called Satake–Tits diagram on which the roots of X are denoted by white circles \circ and those of $\Pi \setminus X$ are denoted by black circles \bullet .

If $\sigma'^(\alpha_i) = \alpha_j$ then it will be indicated by $\overset{i}{\circ} \text{---} \overset{j}{\circ}$ (or $\overset{i}{\bullet} \text{---} \overset{j}{\bullet}$) or by $\overset{i}{\circ} \updownarrow \overset{j}{\circ}$ (or $\overset{i}{\bullet} \updownarrow \overset{j}{\bullet}$) depending on whether these two roots are both in X or in $\Pi \setminus X$ and on their positions on the Dynkin diagram.*

Let us note that the colouring of the vertices is chosen here so that the Satake–Tits diagram for the split form is the Dynkin diagram of its complexified algebra (with black vertices); it differs from that of Helgason ([15], chapter X) where the roots of X are black and those of $\Pi \setminus X$ are white.

As the derived Levi subalgebra \mathfrak{l}' has a unique compact form (up to a conjugation) we have as in the classical case [37, 34].

Theorem 3.2.7 ([2], Théorème 2.8). *The knowledge of the Satake–Tits index determines the real form $\mathfrak{g}_{\mathbb{R}}$ up to an \mathbb{R} -isomorphism.*

3.3. The classification problem and the admissible indices

In practice, the problem is to tell whether such an index comes from an almost split real form of \mathfrak{g} or not. An index coming from a real form will be called admissible. To recognize admissible indices one established in [2] a one-to-one correspondence between conjugacy classes (under $\text{Ad}(G)$ or $\text{Aut}(\mathfrak{g})$) of almost split real forms and those of involutions of the second kind of \mathfrak{g} .

The semi-involution σ' and the standard Cartan semi-involution ω' are assumed to stabilize the same Cartan subalgebra \mathfrak{h} (which is maximally split for σ'). Using an argument going back to Elie Cartan, one proved that there exists a unique (up to $H^{\sigma'}$ -conjugation) Cartan semi-involution which stabilizes \mathfrak{h} and commutes with σ' (see [19] or [28]). By conjugating by H , one may assume that σ' commutes with ω' . Then $\sigma := \sigma' \omega'$ is a \mathbb{C} -linear involution of the second kind of \mathfrak{g} . Hence, $\mathfrak{t} = \mathfrak{h}^{-\sigma}$ is a maximal σ -split toral subalgebra of \mathfrak{g} and the minimal σ' -stable parabolic subalgebra $\mathfrak{p}^+(X)$ is σ -split, within the meaning of Kac and Wang [20], and it is minimal for this property. In particular, the standardization $(\mathfrak{h}, \mathfrak{b}^+)$ is a split pair for σ and thus the restriction of σ to $\mathfrak{l}' (= \mathfrak{g}_X)$ is trivial, cf ([20], section 5) or ([2], section 4).

The main result of Kac and Wang on involutions of the second kind is that split pairs for σ are G^{σ} -conjugate ([20], 5.32). Consequently, we have the following result:

Theorem 3.3.1 ([2], Théorème 4.4). *Consider*

- (1) *the semi-involutions of the first kind σ' of \mathfrak{g} ,*
- (2) *the involutions of the second kind σ of \mathfrak{g} ,*
- (3) *the relation $\sigma' \sim \sigma$ iff*
 - (a) *$\sigma'\sigma = \sigma\sigma'$ is a Cartan semi-involution,*
 - (b) *σ' and σ stabilize a same Cartan subalgebra \mathfrak{a} ,*
 - (c) *\mathfrak{a} is contained in a minimal σ' -stable parabolic subalgebra \mathfrak{p} of \mathfrak{g} .*

Then this relation induces a one-to-one correspondence between conjugacy classes under $Ad(G)$ (resp. $Aut(\mathfrak{g})$) of semi-involutions of the first kind and those of involutions of the second kind of \mathfrak{g} .

Remark 3.3.2. The condition (c) in the theorem is equivalent to require that $\mathfrak{a}^{\sigma'}$ is a maximally split Cartan subalgebra of $\mathfrak{g}^{\sigma'}$.

The involution of the second kind σ acts as the Cartan semi-involution $\sigma\sigma'$ on the real form $\mathfrak{g}_{\mathbb{R}}$ corresponding to σ' , it is called the *Cartan involution* of $\mathfrak{g}_{\mathbb{R}}$, and we have the corresponding *Cartan decomposition*:

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}.$$

In ([9], proposition 2.3), we proved that the Cartan involutions of $\mathfrak{g}_{\mathbb{R}}$ (or, equivalently, the Cartan semi-involutions of \mathfrak{g} commuting to σ') are $G_{\mathbb{R}}$ -conjugate.

Note that the statement of the theorem above does not hold for the semi-involutions of the second kind and the involutions of the first kind, contrarily to what was mentioned in [4] and reproduced in [39]. However, in [7] (corollaire 7.7) we proved a result, in the affine case, which relates the conjugate classes of semi-involutions of the second kind to those of the involutions of the first kind.

3.4. Involutions of the second kind

Let σ be an involution of the second kind of \mathfrak{g} . We may assume that $(\mathfrak{h}, \mathfrak{b}^+)$ is a split pair, so there exists $X \subset \Pi$ of finite type such that $\sigma(\Delta^+) \cap \Delta^+ = \Delta_X^+$ and thus $\mathfrak{p}^+(X)$ is a minimal σ -split parabolic subalgebra. Then σ can be written in this way (see [20], (4.39)):

$$\sigma = \tau\omega Ad(n_X)Ad(s)$$

where

- (i) τ is an involutive diagram automorphism stabilizing X and ω is the standard Cartan involution.
- (ii) $n_X \in N^\tau \cap N^\omega \cap G_X$ such that its image in W_X is the longest element $w_0(X)$, where G_X is the subgroup of G generated by U_α , $\alpha \in \Delta_X$, and W_X the subgroup of W generated by $r_i, i \in X$.
- (iii) $s \in H^\sigma$.

To determine $Ad(s)$, one needs to know the restriction of $Ad(n_X)$ on \mathfrak{g}_X and the action of $Ad(n_X)^2$ on \mathfrak{g} . From the condition $\sigma|_{\mathfrak{g}_X} = 1$, one gets that $w_0(X)$ acts on \mathfrak{h}_X ($:= \mathfrak{h} \cap \mathfrak{g}_X$) as $(-\tau)$ (i.e. $\tau|_{\mathfrak{h}_X} = -w_0(X)$).

Table 1. The longest element w_0 and the sum of positive coroots $2\rho^\vee$ for simple Lie algebras.

Name	Dynkin diagram	w_0	$2\rho^\vee$
$A_n, n \geq 1$		$(r_1 r_2 \dots r_n)(r_1 r_2 \dots r_{n-1}) \dots (r_1 r_2) r_1$	$n\alpha_1 + 2(n-1)\alpha_2 + \dots + i(n-i+1)\alpha_i + \dots + n\alpha_n$
$B_n, n \geq 3$		$(r_1 r_2 \dots r_n)^n$	$2n\alpha_1 + \dots + i(2n-i+1)\alpha_i + \dots + (n-1)(n+2)\alpha_{n-1} + \frac{n(n+1)}{2}\alpha_n$
$C_n, n \geq 2$		$(r_1 r_2 \dots r_n)^n$	$(2n-1)\alpha_1 + \dots + i(2n-i)\alpha_i + \dots + (n-1)(n+1)\alpha_{n-1} + n^2\alpha_n$
$D_n, n \geq 4$		$(r_1 r_2 \dots r_n)^{n-1}$	$2(n-1)\alpha_1 + \dots + i(2n-i-1)\alpha_i + \dots + \frac{n(n-1)}{2}(\alpha_{n-1} + \alpha_n)$
E_6		$(r_1 r_2 r_3 r_4 r_5 r_6)^6$	$2(8\alpha_1 + 11\alpha_2 + 15\alpha_3 + 21\alpha_4 + 15\alpha_5 + 8\alpha_6)$
E_7		$(r_1 r_2 r_3 r_4 r_5 r_6 r_7)^9$	$34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7$
E_8		$(r_1 r_2 r_3 r_4 r_5 r_6 r_7 r_8)^{15}$	$2(46\alpha_1 + 68\alpha_2 + 91\alpha_3 + 135\alpha_4 + 110\alpha_5 + 84\alpha_6 + 57\alpha_7 + 29\alpha_8)$
F_4		$(r_1 r_2 r_3 r_4)^6$	$22\alpha_1 + 42\alpha_2 + 30\alpha_3 + 16\alpha_4$
G_2		$(r_1 r_2)^3$	$2(5\alpha_1 + 3\alpha_2)$

3.5. How to choose n_X ?

By using the results of Tits on the word problem [38] or [2] one gives in [2] an explicit formula of n_X and then one shows that the restriction of $\text{Ad}(n_X)$ on \mathfrak{g}_X is equal to that of $\tau\omega$ (see [2], (4.9)). More precisely, for any reduced expression $(r_{i_1} r_{i_2} \dots r_{i_k})$ of the longest element $w_0(X)$ of W_X , the element $n_X := (m_{i_1} m_{i_2} \dots m_{i_k}) \in N^\tau \cap N^\omega \cap G_X$ does not depend on the choice of the reduced expression of $w_0(X)$ and the automorphism $\text{Ad}(n_X)$ coincides with the traditional prolongation of $w_0(X)$ in an automorphism of \mathfrak{g} of order dividing 4 (see [2], (4.7) and (4.8)). Thus, it is shown that

$$\text{Ad}(n_X)^2 = w_0(X)^2 = \exp(i\pi \text{ad}(2\rho_X^\vee))$$

where $2\rho_X^\vee$ is the sum of the positive coroots of Δ_X (see [2], (4.10.3)).

In table 1 we give, for each complex simple Lie algebra, a reduced expression of the longest element w_0 of the corresponding Weyl group, by noting that in the case $A_n, n \geq 1$, we have

$$w_0(X) = (r_1 r_2 \dots r_n) \cdot (r_1 \dots r_{n-1}) \dots (r_1 r_2) \cdot r_1$$

and that in all the other cases the Coxeter number h is even, and thus $w_0(X) = (c)^{\frac{h}{2}}$, where c is a selected Coxeter transformation such that $(c)^{\frac{h}{2}}$ transforms the fundamental chamber into its opposite (cf [12], chapter V, section 6, proposition 2). In the same table, we also give the expression with α_i^\vee (extracted from the ‘Planches de Bourbaki’ [12]) of the sum of the positive coroots $2\rho^\vee$. In our situation, we choose for n_X the product of the elements corresponding to the connected components of X , and those are the products obtained from table 1 if one replaces r_i by $m_i = \exp(e_i) \exp(-f_i) \exp(e_i)$.

3.6. What is $Ad(s)$ then?

Using the fact that $\sigma|_{\mathfrak{g}_X} = 1$, one can write (see the proof of (4.10.4) in [2]) $Ad(s) = \exp\left(i\pi \text{ad}\left(\sum_{j \in \Omega} \epsilon_j (p_j - p_{\tau(j)})\right)\right)$, where Ω is a complete set of representatives of the τ -orbits in $\Pi \setminus X$ and $\epsilon_j \in \mathbb{Q}$ ($j \in \Omega$) such that

$$2\rho_X^\check + \sum_{j \in \Omega} 2\epsilon_j (p_j - p_{\tau(j)}) \in 2\mathbb{Z}P = \bigoplus_{i \in I} 2\mathbb{Z}p_i \quad (3.6.1)$$

here I denotes the index set $\{1, 2, \dots, n\}$ and for any subset X of Π we confuse sometimes X with the subset $I_X := \{i \in I; \alpha_i \in X\}$ of I . Let $j \in \Omega$ and $u \in \mathbb{Z}$, then by conjugating σ by $\exp\left(\frac{i\pi}{2} \text{ad}(u(p_j + p_{\tau(j)}))\right) \in Z_{H^\omega}(\mathfrak{g}_X)$ one can replace ϵ_j by $\epsilon_j + u$ without thus modifying the others ϵ_i , then $\forall j \in \Omega$, the element $\epsilon_j \in \frac{1}{2}\mathbb{Z}$ may be modified modulo \mathbb{Z} . From (3.6.1) one sees that the class of $2\rho_X^\check$ in $\mathbb{Z}P/2\mathbb{Z}P$ has a representative in $(\mathbb{Z}P_{\Pi \setminus X})^{(-\tau)}$, where $P_{\Pi \setminus X} = \{p_i; \alpha_i \notin X\}$. And by the preceding result, for any such representative $2\bar{\rho}_X$, one may modify ϵ_j in order that

$$2\bar{\rho}_X = \sum_{j \in \Omega} 2\epsilon_j (p_j - p_{\tau(j)}).$$

Hence, we have the following results.

Proposition 3.6.1. *Conjugating by $Z_{H^\omega}(\mathfrak{g}_X)$, we may assume $Ad(s) = \exp\left(\frac{i\pi}{2} \text{ad}(2\bar{\rho}_X)\right)$ and this yields*

$$\sigma = \tau \omega Ad(n_X) \exp\left(\frac{i\pi}{2} \text{ad}(2\bar{\rho}_X)\right). \quad (3.6.2)$$

Remark 3.6.2. Any involution of the second kind σ taking the form (3.6.2) and the associated semi-involution of the first kind $\sigma' := \sigma \omega' = \tau \sigma'_n Ad(n_X) \exp\left(\frac{i\pi}{2} \text{ad}(2\bar{\rho}_X)\right)$ are called standard. Any involution (resp. semi-involution) of the second (resp. the first) kind is G -conjugate to a standard one. Some involutions of the second kind (which are not all standard) are considered in [26] by Pati, Parashar and Kaushal to give the corresponding Iwasawa decomposition for some hyperbolic Kac–Moody Lie algebras.

Let σ' be a standard semi-involution of the first kind and $\mathfrak{g}_\mathbb{R}$ the corresponding almost split real form. Then, the involutive diagram automorphism τ induces the $*$ -action of σ' and thus the Satake–Tits index of $\mathfrak{g}_\mathbb{R}$ is entirely determined by the data (τ, X) . Summarizing, we get the following result.

Theorem 3.6.3. *Let $X \subset \Pi$ be of finite type and τ an involutive diagram automorphism stabilizing X . Then, the couple (τ, X) corresponds to an admissible index iff $\tau|_{\mathfrak{h}_X} = -w_0(X)$ and the class of $2\rho_X^\check$ modulo $2\mathbb{Z}P$ has a $(-\tau)$ -fixed representative in $\mathbb{Z}P_{\Pi \setminus X}$. In particular, if τ is trivial (on $\Pi \setminus X$), then $Ad(n_X)$ should be an involution, i.e. $2\rho_X^\check$ has to be in $2\mathbb{Z}P$.*

4. The relative root system

Let σ' be a semi-involution of the first kind and $\mathfrak{g}_\mathbb{R}$ the corresponding almost split real form. We may assume that σ' is standard (see (3.6.2)) and thus $\sigma = \sigma' \omega' = \omega' \sigma'$ is the Cartan involution of $\mathfrak{g}_\mathbb{R}$. Hence, $\mathfrak{h}_\mathbb{R}$ is a σ -stable maximally split Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{a}_\mathbb{R} := \mathfrak{h}_\mathbb{R}^{-\sigma}$ is a σ -stable maximal split toral subalgebra. The group $K := G_\mathbb{R}^\sigma$ is transitive on the set of

σ -stable maximally split Cartan subalgebras (resp. σ -stable maximal split toral subalgebras) of $\mathfrak{g}_{\mathbb{R}}$ (cf [9], proposition 2.6).

Warning. Note that $\mathfrak{a}_{\mathbb{R}}$ is a maximal Abelian subspace of $\mathfrak{p} := \mathfrak{g}_{\mathbb{R}}^{-\sigma}$ (i.e. a Cartan subspace) but, in contrast to the classical case of real semi-simple Lie algebras, an element of \mathfrak{p} is not necessarily $\text{ad}_{\mathfrak{g}_{\mathbb{R}}}$ -diagonalizable; moreover, a maximal Abelian subspace of \mathfrak{p} may be infinite dimensional, hence maximal Abelian subspaces of \mathfrak{p} are not K -conjugate.

For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ we denote by $\alpha' := \alpha|_{\mathfrak{a}_{\mathbb{R}}}$ the restriction of α to $\mathfrak{a}_{\mathbb{R}}$. The (infinite) *relative (or restricted) root system* $\Delta' := \Delta(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}}) = \{\alpha'; \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})\} \setminus \{0\}$ is more surprising, it was studied by Bardy in ([2], section 3) (see also [3], section 6). In our situation, we have explicitly

$$\mathfrak{a}_{\mathbb{R}} = \bigoplus_{i \in \Omega} \mathbb{R}(p_i + p_{\tau i})$$

where Ω is, as in section 3.6, a complete set of representatives of the τ -orbits in $I \setminus I_X$. It follows that the rank of $\mathfrak{g}_{\mathbb{R}}$ is $r = |\Omega|$. Clearly $\alpha'_i = 0$ iff $i \in I_X$, and for $j, k \in I \setminus I_X$, $\Delta' \ni \alpha'_j = \alpha'_k$ iff j and k are in the same τ -orbit. In particular, the set $\Pi' := \{\alpha'_i; i \in \Omega\}$ is a root basis of Δ' . Denote by Δ'_+ (resp. Δ'_-) the set of positive (resp. negative) roots of Δ' with respect to the basis Π' ; then $\Delta' = \Delta'_+ \cup \Delta'_-$ (disjoint union) and $\sigma(\Delta'_+) = \omega'(\Delta'_+) = \Delta'_-$. For $\alpha' \in \Delta'$, the root space $\mathfrak{g}_{\mathbb{R}, \alpha'}$ associated with α' , with respect to adjoint action of $\mathfrak{a}_{\mathbb{R}}$, is finite dimensional.

4.1. The Iwasawa decomposition

Let $\mathfrak{l}_{\mathbb{R}}$ be the centralizer of $\mathfrak{a}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{n}'_{\pm} = \bigoplus_{\alpha' \in \Delta'_{\pm}} \mathfrak{g}_{\mathbb{R}, \alpha'}$, then we have $\mathfrak{g}_{\mathbb{R}} = \mathfrak{n}'_- \oplus \mathfrak{l}_{\mathbb{R}} \oplus \mathfrak{n}'_+$.

Note that $\mathfrak{l}_{\mathbb{R}} = \mathfrak{a}_{\mathbb{R}} \oplus (\mathfrak{k} \cap \mathfrak{l}_{\mathbb{R}})$ and $\sigma(\mathfrak{n}'_-) = \mathfrak{n}'_+$. It follows that $\mathfrak{n}'_- \subset \mathfrak{k} \oplus \mathfrak{n}'_+$. Hence, one can deduce the *Iwasawa decomposition* for the almost split real Kac–Moody algebra $\mathfrak{g}_{\mathbb{R}}$:

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}'_{\pm}.$$

Consider the positive (or negative) parabolic subgroup $P^{\pm} = P^{\pm}(X)$ of the complex Kac–Moody group G associated with the minimal σ' -stable parabolic subalgebra $\mathfrak{p}^{\pm}(X)$. The subgroup P^{\pm} is the normalizer of $\mathfrak{p}^{\pm}(X)$ in G ; it is stable by σ' and is generated by the subgroups H and U_{α} ($\alpha \in \Delta_{\pm}^{\text{re}} \cup \Delta_X$). Let L be the subgroup of P^{\pm} generated by H and U_{α} ($\alpha \in \Delta_X$), and U_{\pm}^X the normal subgroup of P generated by U_{α} ($\alpha \in \Delta_{\pm}^{\text{re}} \setminus \Delta_X$). Note that the subgroups P, L and U_{\pm}^X are σ' -stable and $P^{\pm} = L \ltimes U_{\pm}^X$. Denote by $P_{\mathbb{R}}^{\pm} = (P^{\pm})^{\sigma'}$, $N_{\mathbb{R}}^{\pm} := (U_{\pm}^X)^{\sigma'}$, $L_{\mathbb{R}} = L^{\sigma'}$, $M_{\mathbb{R}} = L_{\mathbb{R}} \cap L^{\omega'} = L_{\mathbb{R}} \cap K$ and A ‘the vector part’ of $H_{\mathbb{R}} := H^{\sigma'}$. Some computations in the complex algebraic reductive group L (see, for example, [21]) show that the following unique decomposition holds for the minimal real parabolic subgroup $P_{\mathbb{R}}^{\pm}$:

$$P_{\mathbb{R}}^{\pm} = L_{\mathbb{R}} \ltimes N_{\mathbb{R}}^{\pm} = M_{\mathbb{R}} A N_{\mathbb{R}}^{\pm} \quad (\text{Langlands decomposition}).$$

Theorem 4.1.1. *The following unique decomposition holds for the almost split real Kac–Moody group $G_{\mathbb{R}}$:*

$$G_{\mathbb{R}} = K A N_{\mathbb{R}}^{\pm} \quad (\text{Iwasawa decomposition}).$$

Proof. From the Iwasawa decomposition of the complex Kac–Moody group G (cf [18] or [19]), one can deduce the following decomposition:

$$G = G^{\omega'} P^+ = G^{\omega'} L U_+^X, \tag{4.1.1}$$

which is conserved by the semi-involution σ' . Moreover, using the fact that $(P^+)^{\omega'} = L^{\omega'}$, one can see that the component of an element $g \in G$ with respect to U_+^X in (4.1.1) is unique.

It follows that $G_{\mathbb{R}} = (G^{\omega'}L)^{\sigma'}N_{\mathbb{R}}^+$. Let $g = ul \in (G^{\omega'}L)^{\sigma'}$, then $g(\mathfrak{a}_{\mathbb{R}}) = u(\mathfrak{a}_{\mathbb{R}})$ is a maximal split toral subalgebra of $\mathfrak{g}_{\mathbb{R}}$ which is stable by ω' (and therefore by σ). Since the maximal split σ -stable toral subalgebras of $\mathfrak{g}_{\mathbb{R}}$ are K -conjugate (cf [9]; proposition 2.6) one can modify g by K (by composing on the left) so that g stabilizes $\mathfrak{a}_{\mathbb{R}}$. By composing again by the relative Weyl group (see section 4.2) which has representatives in the group K (cf [9]; proposition 2.18) we may assume that g fixes $\mathfrak{a}_{\mathbb{R}}$, i.e. $g \in L_{\mathbb{R}} = M_{\mathbb{R}}A$. The same decomposition holds with $N_{\mathbb{R}}^-$ since $\sigma(N_{\mathbb{R}}^+) = N_{\mathbb{R}}^-$ and σ stabilizes $G_{\mathbb{R}}, K$ and A . The uniqueness of the decomposition is clear. \square

N.B. Note that the argument used in the proof of theorem 4.1.1 is in fact available for general symmetrizable Kac–Moody Lie algebras and groups and thus the Iwasawa decomposition holds for the almost split real algebra $\mathfrak{g}_{\mathbb{R}}$ and the corresponding group $G_{\mathbb{R}}$ in the general case.

4.2. The relative Weyl group

The new aspect to note here is that, as in Borcherd’s work [11], a relative simple root α'_i of the basis Π' may be imaginary (in the sense that no reflection corresponds to this root) and that is the case iff the subset $X(i) := X \cup \{\alpha_i, \alpha_{\tau i}\}$ of Π is not of finite type; then all positive integer multiples of α'_i are still roots. In the case where $X(i)$ is of finite type, the relative root α'_i is called a real relative simple root. If α'_i is real, then the (commutative) product of the longest elements of the Weyl groups corresponding to the connected components of $X(i)$ meeting the τ -orbit of i , commutes with σ' and σ and induces on $\mathfrak{a}_{\mathbb{R}}$ a reflection R'_i such that $R'_i(a) = a - \langle \alpha'_i, a \rangle \alpha'^{\check{}}_i$ ($a \in \mathfrak{a}_{\mathbb{R}}$) for some $\alpha'^{\check{}}_i \in \mathfrak{a}_{\mathbb{R}}$ named the *coroot* of α'_i and satisfying $\langle \alpha'_i, \alpha'^{\check{}}_i \rangle = 2$. Actually, $\alpha'^{\check{}}_i$ is the only element of \mathfrak{a} (defined by equations (3.2.2)) satisfying $\langle \alpha'_i, \alpha'^{\check{}}_i \rangle = 2$ and which is a linear combination of $\alpha^{\check{}}_j$, where j runs over the union of the connected components of $X(i)$ meeting the τ -orbit of i .

The relative Weyl group W' is generated by the reflections R'_i such that α'_i is a real simple relative root. It is isomorphic to N'/H' , where N' (resp. H') is the normalizer (resp. the centralizer) of $\mathfrak{a}_{\mathbb{R}}$ in $G_{\mathbb{R}}$, and it is simply transitive on the set of minimal parabolic subalgebras of $\mathfrak{g}_{\mathbb{R}}$ of sign ϵ and containing $\mathfrak{a}_{\mathbb{R}}$.

If α'_i is a real simple relative root, then $\Delta'_+ \cap \mathbb{Z}^+\alpha'_i \subset \{\alpha'_i, 2\alpha'_i\}$ and $2\alpha'_i$ is a root iff $\tau(i) \neq i$ are contained in the same connected component of $X(i)$ or $\tau(i) = i$ and $\Delta_{X(i)}$ has a root with coefficient 2 on α_i .

Remark 4.2.1. If $X = \emptyset$ and $\tau \neq 1$, then $\mathfrak{g}_{\mathbb{R}}$ is quasi-split and $\mathfrak{t}_{\mathbb{R}} := \mathfrak{h}_{\mathbb{R}}^{\tau} = \sum_{i \in I} \mathbb{R}(p_i + p_{\tau i})$ is a maximal split toral subalgebra. In order to avoid any confusion, one denotes by \bar{Y} (instead of Y') the objects relating to the quasi-split form similar to those introduced above, in the general case, for almost split real forms. In particular, the relative root system $\bar{\Delta}$ is $\Delta(\mathfrak{g}, \mathfrak{h}^{\tau})$ and the corresponding relative Weyl group \bar{W} is generated by \bar{R}_i such that the τ -orbit \bar{i} of i is of finite type. If $J \subset I$ is of finite type and τ -stable, we denote by $\bar{J} := J/(\tau)$ and $\bar{W}_{\bar{J}}$ the subgroup of \bar{W} generated by $\bar{R}_j, j \in \bar{J}$. For $i \in \bar{I}$, set $\bar{\alpha}_i^{\wedge} := \sum_{j \in i} \alpha^{\check{}}_j \in \mathfrak{t}_{\mathbb{R}}$; then $\bar{\alpha}_i$ is real iff $\langle \bar{\alpha}_i, \bar{\alpha}_i^{\wedge} \rangle = \langle \alpha_i, \bar{\alpha}_i^{\wedge} \rangle$ is positive (=1 or 2).

Return now to the general case and put $\alpha_i^{\wedge} = \alpha'^{\check{}}_i$ (resp. $\frac{1}{2}\alpha'^{\check{}}_i$) if α'_i is a real simple relative root such that $2\alpha'_i \notin \Delta'$ (resp. $2\alpha'_i \in \Delta'$) and $\alpha_i^{\wedge} = \sum_{\bar{w} \in \bar{W}_{\bar{X}}} \bar{w}(\bar{\alpha}_i^{\wedge})$ if α'_i is an imaginary simple relative root. Clearly $\alpha_i^{\wedge} \in \mathfrak{a}_{\mathbb{R}}$ in the two cases. If α'_i is a real simple relative root then $\alpha'^{\check{}}_i = \bar{p}_i - R'_i(\bar{p}_i)$, where $\bar{p}_i \in \mathbb{R}(p_i + p_{\tau i}) \subset \mathfrak{a}_{\mathbb{R}}$ such that $\langle \alpha'_i, \bar{p}_i \rangle = 1$ and R'_i is the reflection corresponding to α'_i . Thus, the expression of α_i^{\wedge} results easily from that of $\alpha'^{\check{}}_i$ in this case.

A relative simple root α'_i is imaginary if and only if $\langle \alpha'_i, \alpha'_i \rangle \leq 0$. Here now are some indications to calculate α'_i when α'_i is an imaginary simple relative root and the relative rank $r \geq 2$ (we do not calculate α'_i when $r = 1$). We will assume that α'_i is imaginary of affine type (i.e. $X(i)$ is of affine type) and so $\langle \alpha'_i, \alpha'_i \rangle = 0$. Note that this assumption is always checked for the hyperbolic Kac–Moody Lie algebras. If the diagram automorphism τ (defining the $*$ -action of σ') is non-trivial, then by considering the subset $\bar{X}(\bar{i})$ in $\bar{\Pi}$ (which is also of affine type) one can reduce the problem to the case $\tau = 1$. Assume now that $\tau = 1$ and let $K_{X(i)}$ be the canonical central element of the affine Lie algebra $\mathfrak{g}_{X(i)}$, generated by $\{e_\alpha, f_\alpha, \alpha \in X(i)\}$ (see [17]; chapter 6). Then one can write $K_{X(i)} = \sum_{\alpha \in X(i)} a_\alpha \alpha^\vee$, with positive integers a_α ($\alpha \in X(i)$). In particular, we have $\alpha'_i = \frac{1}{a_{\alpha'_i}} (K_{X(i)} - \sum_{\alpha \in X} a_\alpha \alpha^\vee)$. One can see easily that $\sum_{w \in W_X} w(\alpha^\vee) = 0$ for $\alpha \in X$, and since the canonical central element $K_{X(i)}$ is fixed by W_X (viewed as a subgroup of the affine Weyl group $W_{X(i)}$) we get

$$\alpha'_i = \frac{|W_X|}{a_{\alpha'_i}} K_{X(i)}.$$

4.3. The relative Kac–Moody matrix

In ([2], section 3), Bardy associated with the relative root system Δ' a relative Kac–Moody matrix $B' := (\langle \alpha'_j, \alpha'_i \rangle) = (b'_{i,j})$ which satisfies

$$\begin{aligned} b'_{i,j} &\in \mathbb{Z}, \\ b'_{i,i} &\leq 2, \\ b'_{i,j} &\in \mathbb{Z}^- \text{ for } i \neq j, \\ b'_{i,j} &= 0 \text{ iff } b'_{j,i} = 0. \end{aligned}$$

Theorem 4.3.1 ([2; Théorème 3.10]). *The relative root system is the only subset Δ' of $\bigoplus_{i \in I'} \mathbb{Z}\alpha'_i$, where $I' = (I \setminus I_X)/(\tau) \simeq \Omega$, such that*

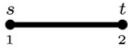
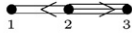
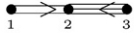

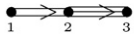


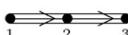
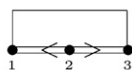
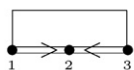
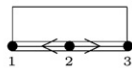
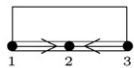
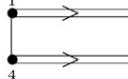
- (1) $\Delta' = \Delta'_+ \cup \Delta'_-$ where $-\Delta'_- = \Delta'_+ = \Delta' \cap (\bigoplus_{i \in I'} \mathbb{Z}^+ \alpha'_i)$.
- (2) $\mathbb{Z}^+ \alpha'_i \cap \Delta'_+ = \begin{cases} \{\alpha'_i\} & \text{if } b'_{i,i} = 2 \\ \{\alpha'_i, 2\alpha'_i\} & \text{if } b'_{i,i} = 1 \\ \mathbb{N}^* \alpha'_i & \text{if } b'_{i,i} \leq 0. \end{cases}$
- (3) For all $\alpha' \in \Delta'_+ \setminus \Pi'$, there exists $i \in I'$ such that $\alpha' - \alpha'_i \in \Delta'_+$.
- (4) For all $i \in I'$ and $\alpha' \in \Delta'_+ \setminus \mathbb{N}\alpha'_i$, the root subset $\Delta' \cap (\alpha' + \mathbb{Z}\alpha'_i)$ is equal to
 - (i) the string $\{\alpha' - p\alpha'_i, \dots, \alpha' + q\alpha'_i\}$ with $p, q \in \mathbb{N}$ such that $p - q = (\frac{2}{b'_{i,i}})\langle \alpha', \alpha'_i \rangle$ if $b'_{i,i} > 0$,
 - (ii) $\{\alpha'\}$ if $b'_{i,i} \leq 0$ and $\text{supp}(\alpha')$ and $\{\alpha'_i\}$ are not linked (with respect to B'),
 - (iii) a set containing $\alpha' + \mathbb{N}\alpha'_i$ if $b'_{i,i} \leq 0$ and $\text{supp}(\alpha')$ and $\{\alpha'_i\}$ are linked.

Remark 4.3.2. Actually when α'_i is an imaginary relative simple root (i.e. $b'_{i,i} \leq 0$), the relative root system Δ' does not depend on the $\langle \alpha'_j, \alpha'_i \rangle = b'_{i,j}$ ($j \in I'$) but only on the fact they are zero or not.

4.4. The relative Dynkin diagram

The normalized relative Kac–Moody matrix associated with B' is the matrix $A' = (a'_{i,j})$ such that $a'_{i,j} = 2(b'_{i,i})^{-1}b'_{i,j} = \langle \alpha'_j, \alpha'_i \rangle$ if $b'_{i,i} > 0$ and $a'_{i,j} = b'_{i,j}$ otherwise (see [3]; section 2). To the relative Kac–Moody matrix is associated a graph $S(B')$, with $|I'|$ vertices, called the relative Dynkin diagram as follows: we associate with each i a vertex equipped with a cross

Table 2. List of strictly hyperbolic symmetrizable Dynkin diagrams.

	$H_{(s,t)}$ 	$\begin{cases} s \leq t \in \mathbb{Z}^+ \\ st > 4 \end{cases}$
BG_3 	CG'_3 	
BG'_3 	CG_3 	
GG_3 	$G'G'_3$ 	
	$G'G_3$ 	
$AD_3^{(2)}$ 	$AC_2^{(1)}$ 	
AGG_3 	$AG'G'_3$ 	
	$AC_3^{(1)}$ 	

if $b'_{i,i} = 1$, with a sign $-$ if $b'_{i,i} < 0$ and with 0 if $b'_{i,i} = 0$. Two vertices i and j are linked iff $b'_{i,j} < 0$; more precisely, if $b'_{i,i}$ and $b'_{j,j}$ are positive and $a'_{i,j}a'_{j,i} = n_{i,j} \leq 4$, the vertices i and j are joined by $m_{i,j} [:= \max(|a'_{i,j}|, |a'_{j,i}|)]$ line(s) with an arrow pointing towards the vertex i if $|a'_{i,j}| > 1$. If $n_{i,j} \geq 5$ or one of the two coefficients $b'_{i,i}$ or $b'_{j,j}$ is non-positive, the vertices i and j are joined by a thick line on which we write $|a'_{i,j}|$ (beside the vertex i) and $|a'_{j,i}|$ (beside the vertex j).

Concerning the nomenclature, the name Z_- stands for \bullet_- . The relative Kac–Moody matrix $\begin{pmatrix} a & -s \\ -t & b \end{pmatrix}$, with $a \geq b$ and $s \leq t$ if $a = b$, is named $H_{(s',t')}^{a,b}$, where $s' = 2s$ (resp. $t' = 2t$) if $a = 1$ (resp. $b = 1$) and $s' = s$ (resp. $t' = t$) otherwise. Actually, for the numbers a, b , the number 2 is omitted, 1 is replaced by \times and (for the Dynkin diagram) a strictly negative number is replaced by $-$. Some other names are given directly in the tables, they are inspired by the notation in tables 2 and 3(a)–(c).

5. Classification

Since the list of hyperbolic Kac–Moody Lie algebras seems to be very long (at least 136) we shall here restrict ourselves to the subclass of the strictly hyperbolic Kac–Moody Lie algebras and the (canonical) hyperbolic Lorentzian extensions of affine Lie algebras (see definition below) to classify their almost split real forms. The classification in the affine case was done in [2].

In table 2 we give, with their names, the list of all strictly hyperbolic symmetrizable Dynkin diagrams. The nomenclature is inspired from the book of Kac ([17], exercise 4.3). It is noted that the strictly hyperbolic Dynkin diagram $G'G_3$ is missed in [33].

Table 3. Hyperbolic Lorentzian extensions of affines Lie algebras.

(a)	
$HA_n^{(1)}$, $2 \leq n \leq 7$, $HB_n^{(1)}$, $3 \leq n \leq 8$, $HC_n^{(1)}$, $2 \leq n \leq 4$, $HD_n^{(1)}$, $4 \leq n \leq 8$,	$HA_{2n}^{(2)}$, $2 \leq n \leq 4$, $HA_{2n-1}^{(2)}$, $3 \leq n \leq 8$, $HD_{n+1}^{(2)}$, $2 \leq n \leq 4$, $H'A_{2n}^{(2)}$, $2 \leq n \leq 4$,
(b)	
$H^2A_1^{(1)}$ $H^3A_1^{(1)}$ $H^2A_2^{(2)}$ $H'^2A_2^{(2)}$ $H^3A_2^{(2)}$ $H'^3A_2^{(2)}$ $H^2A_n^{(1)}$, $n=2, 3$ $H^3A_2^{(1)}$ $H^2B_n^{(1)}$, $n=3, 4$ $H^2A_{2n-1}^{(2)}$, $n=3, 4$ $H^2C_2^{(1)}$ $H^2A_4^{(2)}$ $H^2D_4^{(1)}$ $H^2G_2^{(1)}$ $H^3G_2^{(1)}$ $H^2D_4^{(3)}$ $H^3D_4^{(3)}$	$H^{2*}A_1^{(1)}$ $H^{3*}A_1^{(1)}$ $H^{2*}A_2^{(2)}$ $H'^{2*}A_2^{(2)}$ $H^{3*}A_2^{(2)}$ $H'^{3*}A_2^{(2)}$ $H^{2*}A_n^{(1)}$, $n=2, 3$ $H^{3*}A_2^{(1)}$ $H^{2*}B_n^{(1)}$, $n=3, 4$ $H^{2*}A_{2n-1}^{(2)}$, $n=3, 4$ $H^{2*}C_2^{(1)}$ $H^{2*}A_4^{(2)}$ $H^{2*}D_4^{(1)}$ $H^{2*}G_2^{(1)}$ $H^{3*}G_2^{(1)}$ $H^{2*}D_4^{(3)}$ $H^{3*}D_4^{(3)}$

Table 3. (Continued)

(c)

$H^{(1)}A_2^{(2)}$		$H^{(1)}A_1^{(1)}$		$H^{(1)}A_2^{(2)}$	
$H^{(2)}A_2^{(2)}$		$H^{(2)*}A_2^{(2)}$		$H^{(2)*}A_2^{(2)}$	

Table 4. List of almost split real forms of strictly hyperbolic symmetrizable Kac–Moody algebras.

Name	Satake–Tits index	Relative diagram	Name
$H_{(s,t),2}$			$H_{(s,t)}$
$H_{(2s,t),1}^3$			Z_-
${}^2H_{(s,s),1}$			Z_-
$BG_{3,3}$			BG_3
$BG_{3,2}^3$			$H_{(2,3)}$
$BG_{3,1}^{10}$			Z_-
$BG_{3,1}^{14}$			Z_-
$CG'_{3,3}$			CG'_3
$CG'_{3,1}^{14}$			Z_-
$CG'_{3,1}^6$			Z_-
$BG'_{3,3}$			BG'_3
$BG_{3,2}^3$			$H_{(1,6)}$
$BG_{3,1}^{10}$			Z_-
$BG'_{3,1}^{14}$			Z_-
$CG_{3,3}$			CG_3
$CG_{3,1}^{14}$			Z_-
$CG_{3,1}^6$			Z_-

5.1. Hyperbolic Lorentzian extensions of affine Lie algebras

Consider the Dynkin diagram of an affine Lie algebra of type $X_n^{(k)}$ with the vertices $\alpha_0, \alpha_1, \dots, \alpha_{n_k}$. One extends the Dynkin diagram of $X_n^{(k)}$ by adding a root α_{-1} which is linked only to the root α_0 (i.e. $\langle \alpha_{-1}, \alpha_0 \rangle$ and $\langle \alpha_0, \alpha_{-1} \rangle$ are negative integers, and $\langle \alpha_{-1}, \alpha_i \rangle = 0 \forall i = 1, 2, \dots, n_k$). Let A be the generalized Cartan matrix corresponding

Table 4. (Continued.)

Name	Satake-Tits index	Relative diagram	Name
$GG_{3,3}$			GG_3
$GG_{3,1}^{14}$			Z_-
$GG_{3,1}^6$			Z_-
${}^2GG_{3,2}$			$H_{(1,6)}$
${}^2GG_{3,1}^3$			Z_-
$G'G_{3,3}$			$G'G_3$
$G'G_{3,1}^{14}$			Z_-
$G'G_{3,1}^6$			Z_-
$G'G_{3,1}^{14*}$			Z_-
$G'G'_{3,3}$			$G'G'_3$
$G'G'_{3,1}^{14}$			Z_-
$G'G'_{3,1}^6$			Z_-
${}^2G'G'_{3,2}$			$H_{(2,3)}$
${}^2G'G'_{3,1}^3$			Z_-
$AC_{2,3}^{(1)}$			$AC_2^{(1)}$
$AC_{2,2}^{(1)3}$			$H_{(4,4)}$
$AC_{2,1}^{(1)10}$			Z_-
${}^2AC_{2,2}^{(1)}$			$H_{(2,4)}^\times$
${}^2AC_{2,1}^{(1)3}$			Z_-
${}^2AC_{2,1}^{(1)8}$			Z_-
$AD_{3,3}^{(2)}$			$AD_3^{(2)}$

Table 4. (Continued.)

Name	Satake-Tits index	Relative diagram	Name
${}^2AD_{3,2}^{(2)}$			$H_{(1,8)}^\times$
${}^2AD_{3,1}^{(2)3}$			Z_-
${}^2AD_{3,1}^{(2)8}$			Z_-
$AGG_{3,3}$			AGG_3
$AGG_{3,1}^{14}$			Z_-
${}^2AGG_{3,2}$			$H_{(1,12)}^\times$
${}^2AGG_{3,1}^3$			Z_-
${}^2AGG_{3,1}^8$			Z_-
$AG'G'_{3,3}$			$AG'G'_3$
$AG'G'_{3,1}^{14}$			Z_-
${}^2AG'G'_{3,2}$			$H_{(3,4)}^\times$
${}^2AG'G'_{3,1}^3$			Z_-
${}^2AG'G'_{3,1}^8$			Z_-

Table 4. (Continued.)

Name	Satake-Tits index	Relative diagram	Name
$AC_{3,4}^{(1)}$			$AC_3^{(1)}$
$AC_{3,1}^{(1)21}$			Z_-
${}^2AC_{3,2}^{(1)}$			$H_{(2,4)}^{\times \times}$
${}^2AC_{3,1}^{(1)8}$			Z_-
${}^2AC_{3,1}^{(1)8*}$			Z_-

to this extended Dynkin diagram, \bar{A} the Cartan matrix corresponding to the finite-type Dynkin diagram with the vertices $\alpha_1, \alpha_2, \dots, \alpha_{n_k}$. We denote by m the positive integer $\langle \alpha_{-1}, \alpha_0 \rangle \times \langle \alpha_0, \alpha_{-1} \rangle$, then we have

Proposition 5.1.1. *The generalized Cartan matrix A is symmetrizable, non-singular and Lorentzian (i.e. the signature of the corresponding symmetrized matrix is $(+ + \dots + -)$) and $\det(A) = -m \times \det(\bar{A}) < 0$.*

Proof. The generalized Cartan matrix A is clearly symmetrizable. Let \mathfrak{g} be the corresponding symmetrizable Kac–Moody Lie algebra over the real field \mathbb{R} , $\mathfrak{h} := \mathbb{R}\alpha_{-1} \oplus \mathbb{R}\alpha_0 \oplus \dots \oplus \mathbb{R}\alpha_{n_k}$ the standard Cartan subalgebra and (\cdot, \cdot) the invariant bilinear form on \mathfrak{g} (see (2.4)). We have to show that the symmetric bilinear form (\cdot, \cdot) has signature $(+ + \dots + -)$ on \mathfrak{h} . Let K be the canonical central element of the affine Lie algebra of type $X_n^{(k)}$ above (viewed as a subalgebra of \mathfrak{g}). Then one can see easily that

$$\mathfrak{h} = (\mathbb{R}\alpha_{-1} \oplus \mathbb{R}K) \perp (\mathbb{R}\alpha_1 \oplus \dots \oplus \mathbb{R}\alpha_{n_k})$$

where the orthogonal direct sum is taken relatively to the symmetric bilinear form (\cdot, \cdot) on \mathfrak{h} . As the Dynkin diagram corresponding to the vertices $\alpha_1, \alpha_2, \dots, \alpha_{n_k}$ is of finite type, the symmetric bilinear form (\cdot, \cdot) is positive definite on $(\mathbb{R}\alpha_1 \oplus \dots \oplus \mathbb{R}\alpha_{n_k})$. Since $(K, K) = 0$ and $\langle \alpha_{-1}, K \rangle = \langle \alpha_{-1}, \alpha_0 \rangle < 0$, the matrix of the bilinear form (\cdot, \cdot) on $(\mathbb{R}\alpha_{-1} \oplus \mathbb{R}K)$, with respect to the basis (α_{-1}, K) , takes the form $\begin{pmatrix} a & -b \\ -b & 0 \end{pmatrix}$ where a and b are two positive real numbers. In particular, the bilinear form (\cdot, \cdot) has signature $(+ -)$ on $(\mathbb{R}\alpha_{-1} \oplus \mathbb{R}K)$. Hence,

Table 5. List of almost split real forms for the canonical hyperbolic Lorentzian extensions of exceptional affine Kac–Moody algebras.

Name	Satake-Tits index	Relative diagram	Name
$HA_{1,3}^{(1)}$			$HA_1^{(1)}$
$HA_{1,2}^{(1)3}$			$H_{(1,2)}^0$
$HG_{2,4}^{(1)}$			$HG_2^{(1)}$
$HG_{2,2}^{(1)14}$			$H_{(1,12)}^0$
$HF_{4,6}^{(1)}$			$HF_4^{(1)}$
$HF_{4,3}^{(1)21}$			$H^{2*}A_2^{(2)\times}$
$HF_{4,2}^{(1)36}$			$H_{(1,2^4,4!)}^0$
$HF_{4,2}^{(1)52}$			$H_{(1,2^7,3^2)}^0$

the symmetric bilinear form (\cdot, \cdot) is nondegenerate on \mathfrak{h} and has signature $(+\dots+ -)$. The last formula for the determinant holds by describing the matrix A using \bar{A} . \square

Definition 5.1.2. *The Kac–Moody Lie algebra \mathfrak{g} associated with the extended Dynkin Diagram above is called a Lorentzian extension of the affine Lie algebra $X_n^{(k)}$. The positive integer $m (= \langle \alpha_{-1}, \alpha_0^\vee \rangle \times \langle \alpha_0, \alpha_{-1}^\vee \rangle)$ is called the link multiplicity of the Lorentzian extension of $X_n^{(k)}$.*

5.1.1. Remarks and notation

- (1) Note that if the Kac–Moody algebra \mathfrak{g} (Lorentzian extension of the affine Lie algebra $X_n^{(k)}$) is hyperbolic, then the link multiplicity $m \in \{1, 2, 3, 4\}$ since the Dynkin diagram corresponding to the vertices α_{-1} and α_0 (which will be denoted as $D_{-1,0}$) should be of finite or affine type.
- (2) The Lorentzian extension of an affine Lie algebra of type $X_n^{(k)}$ for which the two vertices α_{-1} and α_0 are simply linked (i.e. $m = 1$) was considered by Henneaux and Julia in [16] (see also [17]; section 5.11) where they denoted it by $X_n^{(k)\wedge}$. They called such a Lorentzian extension the standard or twisted overextension of X_n depending on whether $k = 1$ or $k = 2, 3$. Here, we shall choose the notation $HX_n^{(k)}$ for the Lorentzian extension $X_n^{(k)\wedge}$ when this one is hyperbolic (and $m = 1$) and it will be called the *canonical hyperbolic Lorentzian extension* of $X_n^{(k)}$.
- (3) In the case $m = 2$ or 3 (i.e. the rank-2 Dynkin diagram $D_{-1,0}$ is of type C_2 or G_2) we will denote by $H^m X_n^{(k)}$ (resp. $H^{m*} X_n^{(k)}$) the Lorentzian extension of $X_n^{(k)}$ if this one is hyperbolic and the root α_{-1} is shorter (resp. longer) than α_0 .
- (4) In the case $m = 4$, the rank-2 Dynkin diagram $D_{-1,0}$ is of affine type and there are two possibilities:

Table 5. (Continued.)

Name	Satake-Tits index	Relative diagram	Name
$HE_{6,8}^{(1)}$			$HE_6^{(1)}$
$HE_{6,4}^{(1)28}$			$H^{2*}A_2^{(1)}$
${}^2HE_{6,6}^{(1)}$			$HF_4^{(1)}$
${}^2HE_{6,4}^{(1)15}$			$HA_4^{(2)\times}$
${}^2HE_{6,2}^{(1)45}$			$H_{(1,2^4,4!)}^0$
${}^2HE_{6,2}^{(1)78}$			$H_{(1,2^7,3^2)}^0$
$HE_{7,9}^{(1)}$			$HE_7^{(1)}$
$HE_{7,6}^{(1)9}$			$HF_4^{(1)}$
$HE_{7,5}^{(1)28}$			$HC_3^{(1)}$
$HE_{7,2}^{(1)69}$			$H_{(1,2^5,6!)}^0$
$HE_{7,2}^{(1)133}$			$H_{(1,2^{10},3^4,5,7)}^0$
$HE_{7,1}^{(1)248}$			Z_-
$HE_{8,10}^{(1)}$			$HE_8^{(1)}$
$HE_{8,6}^{(1)28}$			$HF_4^{(1)}$
$HE_{8,2}^{(1)120}$			$H_{(1,2^6,8!)}^0$
$HE_{8,2}^{(1)248}$			$H_{(1,2^{14},3^5,5^2,7)}^0$
$HA_{2,3}^{(2)}$			$HA_2^{(2)}$
$HA_{2,2}^{(2)3}$			$H_{(1,2)}^0$
$H'A_{2,3}^{(2)}$			$H'A_2^{(2)}$
$H'A_{2,1}^{(2)6}$			Z_-

Table 5. (Continued.)

Name	Satake-Tits index	Relative diagram	Name
$HE_{6,6}^{(2)}$			$HE_6^{(2)}$
$HE_{6,3}^{(2)21}$			$HA_2^{(2)\times}$
$HE_{6,2}^{(2)52}$			$H_{(1,2^7,3^2)}^0$
$HD_{4,4}^{(3)}$			$HD_4^{(3)}$
$HD_{4,2}^{(3)14}$			$H_{(1,12)}^0$

Table 6. List of almost split real forms for the canonical hyperbolic Lorentzian extensions of classical affine Kac-Moody algebras.

Name	Satake-Tits index	Parameters	Relative diagram	Name
$HA_{n,n+2}^{(1)}$		$2 \leq n \leq 7$		$HA_n^{(1)}$
${}^4HA_{n,n-2}^{(1)}$		$n = 3$		Z_-
		$n = 4$		$H_{(3,3)}$
		$5 \leq n \leq 7$		$AD_{n-2}^{(2)}$
${}^1HA_{2k-1,k+1}^{(1)}$		$k=2$		$H^{2*}A_1^{(1)}$
		$k=3, 4$		$H^{2*}A_{k-1}^{(1)}$
${}^8HA_{7,1}^{(1)}$				Z_-
${}^8HA_{7,1}^{(1)'}$				Z_-
${}^2HA_{2k-1,k+2}^{(1)}$		$2 \leq k \leq 4$		$HC_k^{(1)}$

Table 6. (Continued.)

Name	Satake-Tits index	Parameters	Relative diagram	Name
${}_{2(k-r)-1}^2 HA_{2k-1,r+2}^{(1)}$		$1 < r < k$		$HA_{2r}^{(2)\times}$
${}_{2(k-r)-1}^2 HA_{2k-1,r+2}^{(1)}$		$2 \leq k \leq 4$		$HA_2^{(2)\times}$
${}_{2(k-r)-1}^2 HA_{2k-1,r+2}^{(1)}$		$r=1$		$H_{(1,2^k,k!)}^0$
${}_{2(k-r)-1}^2 HA_{2k-1,r+2}^{(1)}$		$r=0$		$H^{2*}C_2^{(1)\times}$
${}_{\frac{2}{3}} HA_{2k-1,k}^{(1)'}$		$k=4$		$H'^2 A_2^{(2)\times}$
${}_{\frac{2}{3}} HA_{2k-1,k}^{(1)'}$		$k=3$		$H_{(2,8)}^0$
${}_{\frac{2}{3}} HA_{2k-1,k}^{(1)'}$		$k=2$		$H^{2*}A_1^{(1)\times\times}$
${}_{2(k-r)-1}^2 HA_{2k-1,r}^{(1)}$		$r=3, k=4$		$H_{(1,2^{k+1},(k-2)!)}^0$
${}_{2(k-r)-1}^2 HA_{2k-1,r}^{(1)}$		$r=2 < k \leq 4$		Z_-
${}_{\frac{2}{6}} HA_{2k-1,k-2}^{(1)}$		$k=3$		$H_{(1,2^8,3^2)}^0$
${}_{\frac{2}{6}} HA_{2k-1,k-2}^{(1)}$		$k=4$		Z_-
${}_{\frac{2}{1}} HA_{7,1}^{(1)}$				$H_{(2^4(4!), 2^4(4!))}^{0,0}$
${}_{\frac{2}{7}} HA_{7,2}^{(1)'}$				$HA_{2r}^{(2)\times}$
${}_{2(k-r)}^2 HA_{2k,r+2}^{(1)}$		$1 \leq k \leq 3$		$HA_2^{(2)\times}$
${}_{2(k-r)}^2 HA_{2k,r+2}^{(1)}$		$r=1$		$H_{(1,2^k,k!)}^0$
${}_{2(k-r)}^2 HA_{2k,r+2}^{(1)}$		$r=0$		$H_{(1,2^{k+1})}^0$
${}_{\frac{2}{5-k}} HA_{2k,2}^{(1)}$		$k=2, 3$		$H^{2*}A_1^{(1)\times\times}$
${}_{\frac{2}{3}} HA_{6,3}^{(1)}$				Z_-
${}_{\frac{2}{6}} HA_{6,1}^{(1)}$				

- (i) The Dynkin diagram $D_{-1,0}$ is of type $A_1^{(1)}$: the two roots α_{-1} and α_0 have the same length and we denote by $H^{(1)}X_n^{(k)}$ the Lorentzian extension of $X_n^{(k)}$ if this one is hyperbolic.

Table 6. (Continued.)

Name	Satake-Tits index	Parameters	Relative diagram	Name
$n-rHB_{n,r+2}^{(1)}$		$3 \leq r \leq n$		$HB_r^{(1)}$
		$r=2$		$HC_2^{(1)}$
		$r=1$		$H^{2*}A_1^{(1)}$
		$r=0$		$H_{(1,2^n, n!)}^0$
$nHB_{n,2}^{(1)'}$		$3 \leq n \leq 8$		$H_{(2,2^n, n!)}^0$
$n+1HB_{n,1}^{(1)}$		$3 \leq n \leq 8$	$\overset{1}{\bullet} -$	Z_-
$4HB_{n,n-2}^{(1)''}$		$n=4$		$H_{(1,2^3, 4!)}^0$
		$n=5$		$HA_1^{(1)}$
		$6 \leq n \leq 8$		$HD_{n-3}^{(2)}$
$n-rHB_{n,r-2}^{(1)}$		$r=4 < n \leq 8$		$H_{(1,2^{n-2}, 4!, (n-4)!)}^0$
		$r=5 < n \leq 8$		$HA_1^{(1)}$
		$6 \leq r < n$		$HD_{r-1}^{(2)}$
$8HB_{7,1}^{(1)'}$			$\overset{7}{\bullet} -$	Z_-
$1HB_{8,1}^{(1)}$			$\overset{7}{\bullet} -$	Z_-
$8HB_{8,2}^{(1)''}$				$H_{(2^6, 8!, 2^7, 8!)}^{0,0}$
$8HB_{8,2}^{(1)'''}$				$H_{(2, 2^{14}, 3^5, 5^2, 7)}^0$
$HC_{n,n+2}^{(1)}$		$2 \leq n \leq 4$		$HC_n^{(1)}$
$3HC_{n,n-1}^{(1)}$		$n=2$	$\overset{2}{\bullet} -$	Z_-
		$n=3$		$H_{(1,8)}^\times$
		$n=4$		$H^{2*}A_2^{(2)\times}$

(ii) The Dynkin diagram $D_{-1,0}$ is of type $A_2^{(2)}$: we denote by $H^{(2)}X_n^{(k)}$ (resp. $H^{(2)*}X_n^{(k)}$) the Lorentzian extension of $X_n^{(k)}$ if this one is hyperbolic and the root α_{-1} is shorter (resp. longer) than α_0 .

Table 6. (Continued.)

Name	Satake-Tits index	Parameters	Relative diagram	Name
${}^4HC_{n,n-2}^{(1)}$		$n=3$ $n=4$		Z_- $H_{(1,2^8,3^2)}^0$
${}^1HC_{n,3}^{(1)}$		$n=2$ $n=3, 4$		$H^{2*}A_1^{(1)\times}$ $H^{2*}A_2^{(2)\times}$
${}^1HC_{4,3}^{(1)'}$				$H^{(2)*}A_2^{(2)\times}$
${}^1HC_{4,4}^{(1)}$				$H^{2*}C_2^{(1)}$
${}^nHC_{n,2}^{(1)'}$		$2 \leq n \leq 4$		$H_{(1,2^n,n!)}^0$
${}^{n-2}HC_{n,2}^{(1)}$		$n=3$ $n=4$		$H_{(1,2^4)}^0$ $H_{(1,2^6)}^0$
${}^nHC_{n,2}^{(1)'}$		$n=2$ $n=4$		$H_{(2,2^3)}^0$ $H_{(2^4,4!,2^4,4!)}^{0,0}$
${}^1HD_{n,n+2}^{(1)}$		$4 \leq n \leq 8$		${}^1HD_n^{(1)}$
${}^{n-r}HD_{n,r+2}^{(1)}$		$r=2, n \text{ even}$ $4 \leq n \leq 8$		${}^1HC_2^{(1)}$ ${}^1HB_r^{(1)}$
${}^{n-1}HD_{n,3}^{(1)}$		$n=5, 7$		$H^{2*}A_1^{(1)}$
${}^nHD_{n,2}^{(1)}$		$n=4, 6, 8$		$H_{(1,2^{n-1},n!)}^0$
${}^nHD_{n,2}^{(1)'}$		$n=4, 6, 8$		$H_{(2,2^{n-1},n!)}^0$
${}^1HD_{8,2}^{(1)}$				$H_{(1,2^{11},3^4,5,7)}^0$
${}^8HD_{8,2}^{(1)'}$				$H_{(i,i)}^{0,0}$

Table 6. (Continued.)

Name	Satake-Tits index	Parameters	Relative diagram	Name
$8HD_{8,2}^{(1)''''}$				$H_{(j,j)}^{0,0}$
$n+1HD_{n,1}^{(1)}$		$n=5, 7$		Z_-
$n+1HD_{n,1}^{(1)'}$		$n=6, 7$		Z_-
$4HD_{n,n-2}^{(1)}$		$n=5$		$H''A_1^{(1)}$
		$n=6$		$H'C_2^{(1)}$
		$n=7, 8$		$H'B_{n-4}^{(1)}$
$n-rHD_{n,r-2}^{(1)}$		$n=6$ $r=4$		$H_{(1,2^4,4!)}^0$
		$n=7$ $r=5$		$HA_1^{(1)}$
		$n=8$ $r=4$		$H_{(1,2^5,(4!)^2)}^0$
		$n=8$ $r=6$		$HD_3^{(2)}$
$1HD_{2k,k+2}^{(1)}$		$n=2k$ $k=3, 4$		$HC_k^{(1)}$
$n-rHD_{n,r+2}^{(1)}$		$r=2, n \text{ odd}$		$HC_2^{(1)}$
		$3 \leq r < n$ $n-r \text{ odd}$		$HB_r^{(1)}$
$n-1HD_{n,3}^{(1)}$		$n=4, 6, 8$		$H^{2*}A_1^{(1)}$
$n+1HD_{n,1}^{(1)}$		$n=4, 6, 8$		Z_-
$nHD_{n,2}^{(1)}$		$n=5, 7$		$H_{(1,2^{n-1}.(n-1)!)}^0$

Table 6. (Continued.)

Name	Satake-Tits index	Parameters	Relative diagram	Name
${}^2_n HD_{n,2}^{(1)'}$		$n=5, 7$		$H_{(2,2^{n-1},(n-1)!)}^0$
${}^2_1 HD_{2k+1,k+2}^{(1)}$		$n=2k+1$ $k=2, 3$		$HA_{2k}^{(2)\times}$
${}^2_4 HD_{n,n-3}^{(1)}$		$n=5$		$H_{(1,2^3,4!)}^0$
		$n=6$		$HA_1^{(1)}$
		$n=7, 8$		$HD_{n-4}^{(2)}$
${}^2_3 HD_{n,n-5}^{(1)}$		$n=7$		$H_{(1,2^5,4!)}^0$
		$n=8$		$HA_1^{(1)}$
${}^2_7 HD_{7,1}^{(1)}$			6 -	Z_-
${}^2_8 HD_{8,1}^{(1)}$			7 -	Z_-
${}^{k-r} HA_{2k,r+2}^{(2)}$		$r=0$		$H_{(1,2^k,k!)}^0$
		$r=1$		$HA_2^{(2)}$
		$2 \leq r \leq k$		$HA_{2r}^{(2)}$
${}^3 HA_{2k,k-1}^{(2)'}$		$k=2$	2 -	Z_-
		$k=3$		$H_{(2,4)}^\times$
		$k=4$		$H^2 A_2^{(2)\times}$
${}^1 HA_{8,2}^{(2)}$				$H_{(2,4)}^\times$
${}^{k-2} HA_{2k,1}^{(2)}$		$k=3,4$	2 -	Z_-
${}^4 HA_{2k,k-2}^{(2)'}$		$k=3$	3 -	Z_-
		$k=4$		$H_{(2,2^7,3^2)}^0$
${}^1 HA_{8,1}^{(2)}$			3 -	Z_-
$H' A_{2k,k+2}^{(2)}$		$2 \leq k \leq 4$		$H' A_{2k}^{(2)}$

Table 6. (Continued.)

Name	Satake-Tits index	Parameters	Relative diagram	Name
$4H'A_{2k,k-2}^{(2)}$		$k=3$ $k=4$	$\begin{matrix} 3 & - \\ 1 & \xrightarrow{2^5, 3^2} \\ 4 & \xrightarrow{3} 0 \end{matrix}$	Z_- $H_{(1,2^8, 3^2)}^0$
$kH'A_{2k,2}^{(2)'}$		$k=3$ $k=4$	$\begin{matrix} \times & 8 & \xrightarrow{2^2, (3!)} \\ -1 & \xrightarrow{3} 0 \\ 2^5 (4!) & \xrightarrow{2^3 (4!)} \\ 0 & \xrightarrow{4} 0 \end{matrix}$	$H_{(8,2^2, 3!)}^{\times, 0}$ $H_{(2^3, 4!, 2^5, 4!)}^{0, 0}$
$1H'A_{2k,2}^{(2)}$		$k=2$ $k=3, 4$	$\begin{matrix} 1 & \xrightarrow{8} \times \\ 2 & \xrightarrow{0} \\ \times & 2 & \xrightarrow{4} \times \\ 2 & \xrightarrow{0} \end{matrix}$	$H_{(1,8)}^{\times}$ $H_{(2,4)}^{\times \times}$
$1H'A_{2k,1}^{(2)}$		$2 \leq k \leq 4$	$\bullet -$	Z_-
$k-2H'A_{2k,1}^{(2)'}$		$k=3, 4$	$\bullet -$	Z_-
$3H'A_{4,1}^{(2)}$			$\bullet -$	Z_-
$1H'A_{8,3}^{(2)}$			$\begin{matrix} \times \\ 0 & \xrightarrow{2} 4 \end{matrix}$	$H^{2*}A_2^{(2)\times}$
$3H'A_{8,3}^{(2)}$			$\begin{matrix} \times \\ -1 & \xrightarrow{3} 4 \end{matrix}$	$H^{(2)*}A_2^{(2)\times}$
$1H'A_{8,2}^{(2)'}$			$\begin{matrix} 2^4 (3!) & 1 \\ 0 & \xrightarrow{2} 4 \end{matrix}$	$H_{(1,2^4, 3!)}^0$
$HA_{2k-1,k+2}^{(2)}$		$3 \leq k \leq 8$		$HA_{2k-1}^{(2)}$
$1HA_{2k-1,r+2}^{(2)}$		$3 \leq k \leq 8$	$r=1$ $\begin{matrix} \times \\ -1 & \xrightarrow{0} 2 \end{matrix}$	$HA_2^{(2)\times}$
		$2 < 2r < k$	$\begin{matrix} \times \\ -1 & \xrightarrow{0} 2 & \xrightarrow{4} 2r \end{matrix}$	$HA_{2r}^{(2)\times}$
		$2r = k$	$\begin{matrix} \times \\ -1 & \xrightarrow{0} 2 & \xrightarrow{4} k \end{matrix}$	$HC_r^{(1)}$
$kHA_{2k-1,2}^{(2)}$		$3 \leq k \leq 8$	$\begin{matrix} 1 & \xrightarrow{2^k, (k!)} \\ -1 & \xrightarrow{0} 0 \end{matrix}$	$H_{(1,2^k, k!)}^0$
$4HA_{2k-1,k-2}^{(2)}$		$k=4$ $k=5$	$\begin{matrix} 2 & \xrightarrow{2^2 (4!)} \\ -1 & \xrightarrow{4} 0 \end{matrix}$ $\begin{matrix} \bullet & \xrightarrow{4} 5 \end{matrix}$	$H_{(2,2^2, 4!)}^0$ $H'A_2^{(2)}$
		$6 \leq k \leq 8$	$\begin{matrix} \bullet & \xrightarrow{4} 5 & \xrightarrow{k} \end{matrix}$	$H'A_{2(k-4)}^{(2)}$
$k-6HA_{2k-1,1}^{(2)}$		$k=7, 8$	$\bullet -$	Z_-
$k+1HA_{2k-1,1}^{(2)}$		$k=6, 7$	$\bullet -$	Z_-

Table 6. (Continued.)

Name	Satake-Tits index	Parameters	Relative diagram	Name
$8HA_{15,2}^{(2)'}$				$H_{(2^6, 8!, 2^7, 8!)}^{0,0}$
$1HA_{15,2}^{(2)}$				$H_{(1, 2^{11}, 3^4, 5, 7)}^0$
$8HA_{15,2}^{(2)''}$				$H_{(1, 2^{15}, 3^5, 5^2, 7)}^0$
$n-rHD_{n+1, r+2}^{(2)}$		$2 \leq n \leq 4$	$r=0$ $r=1$ $2 \leq r \leq n$	$H_{(1, 2^n, n!)}^0$ $HA_1^{(1)}$ $HD_{r+1}^{(2)}$
$3HD_{n+1, n-1}^{(2)'}$			$n=3$ $n=4$	$H_{(4, 2^3, 3!)}^{\times, 0}$ $H^{(1)}A_2^{(2)\times}$
$4HD_{n+1, n-2}^{(2)'}$			$n=3$ $n=4$	Z_- $H_{(2, 2^7, 3^2)}^0$
$1HD_{5,2}^{(2)}$			$n=4$	$H_{(4, 2^3, 3!)}^0$
$4HD_{5,2}^{(2)'}$			$n=4$	$H_{(2^4, 4!, 2^4, 4!)}^{0,0}$
$1HD_{5,1}^{(2)}$			$n=3$	Z_-

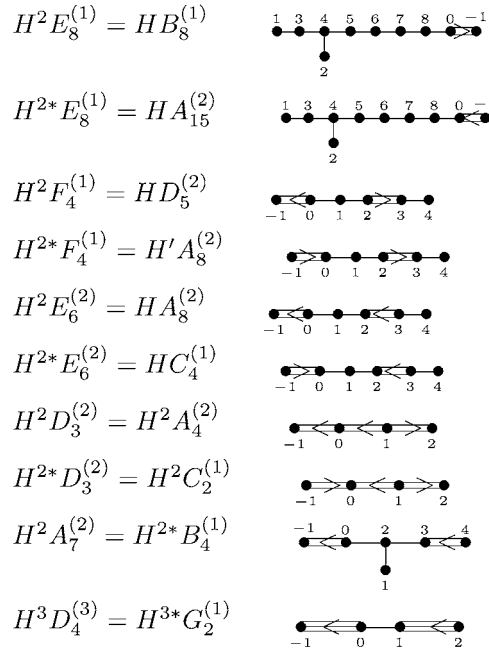
(5) Note that for the twisted Dynkin diagram $A_{2n}^{(2)}$ there exist two conventions for the root α_0 ; so we consider the two corresponding canonical hyperbolic Lorentzian extensions $HA_{2n}^{(2)}$ and $H'A_{2n}^{(2)}$, each one is the dual of the other. We adopt the same notation for the other hyperbolic Lorentzian extensions (with $m = 2, 3, 4$): $H^x A_{2n}^{(2)}$ and $H'^x A_{2n}^{(2)}$, $x = 2, 2^*, 3, 3^*, (1), (2), (2)^*$.

5.1.2. Canonical hyperbolic Lorentzian extensions of affine Lie algebras. It was stated in [16] that all Lorentzian extensions (with $m = 1$) of exceptional affine Lie algebras are hyperbolic (including $HA_1^{(1)}, HA_2^{(2)}, H'A_2^{(2)}$ and $HD_4^{(3)}$); whereas there are some restrictions for those of classical affine Lie algebras.

In table 3(a) we give, with their names, the list of all canonical hyperbolic Lorentzian extensions of classical affine Lie algebras other than $HA_1^{(1)}, HA_2^{(2)}, H'A_2^{(2)}$ and $HD_4^{(3)}$ which we had regarded as exceptional. In particular, the four rank-10 hyperbolic Kac–Moody algebras which are often denoted as E_{10}, BE_{10}, CE_{10} and DE_{10} are Lorentzian extensions of affine algebras and they are denoted here respectively as $HE_8^{(1)}, HB_8^{(1)}, HA_{15}^{(2)}$ and $HD_8^{(1)}$. Note that table 3(a) contains all the hyperbolic Dynkin diagrams of rank from 7 to 10 (see [17]; exercise 4.3 or [33]). Finally, let us announce that the two rank-5 hyperbolic Dynkin diagrams $HC_3^{(1)}$ and $HD_4^{(2)}$ are missed in [33].

5.1.3. *Hyperbolic Lorentzian extensions of affine Lie algebras with $m = 2, 3, 4$.* The hyperbolic Lorentzian extensions of affine Lie algebras with $m = 2, 3$ are given in table 3(b).

The other missed hyperbolic Lorentzian extensions are, if one changes the labelling, canonical or already enumerated with different names on table 3(b). In particular, we have



The hyperbolic Lorentzian extensions of affine Lie algebras with $m = 4$ are given in table 3(c).

5.2. Tables

In the case of the strictly hyperbolic algebras and the canonical Lorentzian extensions of exceptional affine algebras (including $HA_1^{(1)}, HA_2^{(2)}, H'A_2^{(2)}$ and $HD_4^{(3)}$), we indicate successively, in tables 4 and 5, the name of the almost split real form, its Satake–Tits diagram (represented in accordance with (3.2.6)), the conditions on the parameters and the relative root system (with its name and its Dynkin diagram). The name of the real form arises in this case in the form ${}^a Z_{n,r}^m$ where Z_n is the name of the hyperbolic algebra, $r = |I'|$ is the relative rank, $a = 1, 2$ is the order of the diagram automorphism τ defining the *-action of σ' (omitted if it equals 1) and m (omitted if it equals 0, i.e. in the quasi-split case) is the dimension of the derived Levi subalgebra $l' = \mathfrak{g}_X$ (cf (3.2.5)).

In the case of the canonical hyperbolic Lorentzian extensions of classical affine algebras, we indicate successively, in table 6, the name of the almost split real form, its Satake–Tits diagram (represented in accordance with (3.2.6)), the conditions on the parameters and the relative root system (with its name and its Dynkin diagram). The name of the real form arises in this case in the form ${}^a_i Z_{n,r}$ where Z_n is the name of the hyperbolic algebra, a and r are given according to same conventions as above, and i (optional) is a parameter: it is often the cardinal of the smallest connected component of X .

The determination of all admissible indices is made using (3.6.3). Actually, when the relative rank is at least 2, this can also be deduced from proposition 2.11 in [2] and the known lists of admissible indices for simple or affine Lie algebras (see [37] or [35] and [2]). It is easy

to determine relative Dynkin diagram using the rules explained in section 3 to compute α'_i or α''_i ; the results already known for semi-simple or affine Lie algebras may also be used. We do not calculate $a'_{i,i}$ in rank 1 (i.e. when $b'_{i,i} < 0$).

Finally, let us note (as it can be shown in the tables above) that for the strictly hyperbolic almost split real forms, of rank ≥ 2 , the corresponding relative Dynkin diagrams are also strictly hyperbolic; whereas, for the hyperbolic canonical Lorentzian extensions of affine algebras, the relative Dynkin diagrams, associated with their almost split real forms of rank ≥ 3 , are hyperbolic but they are not always Lorentzian extensions of affine diagrams.

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References

- [1] Andruskiewitsch N 1992 Some forms of Kac–Moody algebras *J. Algebra* **147** 324–44
- [2] Back-Valente V, Bardy-Panse N, Ben Messaoud H and Rousseau G 1995 Formes presque déployées d’algèbres de Kac–Moody, Classification et racines relatives *J. Algebra* **171** 43–96
- [3] Bardy-Panse N 1996 Systèmes de racines infinis *Mémoire de la S.M.F.* vol 65
- [4] Batra P 2000 Invariants of real forms of affine Kac–Moody Lie algebras *J. Algebra* **223** 208–36
- [5] Bausch J 1988 Etude et classification des automorphismes d’ordre fini et de première espèce des algèbres de Kac–Moody affines *Revue de l’Institut Elie Cartan, Nancy* **11** 5–124
- [6] Bausch J and Rousseau G 1988 Involutions de première espèce des algèbres de Kac–Moody affines *Revue de l’Institut Elie Cartan, Nancy* **11** 125–39
- [7] Ben Messaoud H and Rousseau G 2003 Classification des formes réelles presque compactes des algèbres de Kac–Moody affines *J. Algebra* **267** 443–513
- Ben Messaoud H and Rousseau G 2004 Coquilles corrigées dans *J. Algebra* **279** 409–12
- [8] Ben Messaoud H and Rousseau G 2007 Sous-algèbres de Cartan des algèbres de Kac–Moody affines réelles presque compactes *J. Lie Theory* **17** (1) 001–025
- [9] Ben Messaoud H and Rousseau G 2006 Sous-algèbres de Cartan des algèbres de Kac–Moody réelles presque déployées *J. Math. Soc. Japan* **58** (4) 1009–1030
- [10] Berman S and Pianzola A 1987 Generators and relations for real forms of some Kac–Moody Lie algebras *Commun. Algebra* **15** 935–59
- [11] Borcherds R E 1988 Generalized Kac–Moody algebras *J. Algebra* **115** 501–12
- [12] Bourbaki N 1981 *Groupes et algèbres de Lie* chapters 4–6 (Paris: Masson)
- [13] Damour T, Henneaux M, Julia B and Nicolai H 2001 Hyperbolic Kac–Moody algebras and chaos in Kaluza–Klein models *Phys. Lett. B* **509** 323
- [14] Goddard P and Olive D I 1984 Algebras, lattices and strings *Vertex Operators in Mathematics and Physics (MSRI Publication vol 3)* (Berlin: Springer) p 51
- [15] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)
- [16] Henneaux M and Julia B 2003 Hyperbolic billiards of pure $D = 4$ supergravities *J. High Energy Physics* JHEP05(2003)47
- [17] Kac V G 1990 *Infinite Dimensional Lie algebras* 3rd edn (Cambridge: Cambridge University Press)
- [18] Kac V G and Peterson D H 1985 Defining relations of certain infinite dimensional groups *Élie Cartan et les mathématiques d’aujourd’hui (Lyon, 1984) (Astérisque no hors série)* pp 165–208
- [19] Kac V G and Peterson D H 1987 On geometric invariant theory of infinite dimensional groups *Algebraic Groups (Utrecht, 1986) (Springer Lecture Note in Math. vol 1271)* pp 109–42
- [20] Kac V G and Wang S P 1992 On automorphisms of Kac–Moody algebras and groups *Adv. Math.* **92** 129–95
- [21] Knapp A W Lie groups beyond an introduction *Prog. in Math.* vol 140 2nd edn
- [22] Lepowsky J and Moody R V 1979 Hyperbolic Lie algebras and quasi-regular cusps on Hilbert modular surfaces *Math. Ann.* **245** 63
- [23] Levstein F 1988 A classification of involutive automorphisms of an affine Kac–Moody Lie algebra *J. Algebra* **114** 489–518

- [24] Matthias R G, Olive D I and West P C 2002 A class of Lorentzian Kac–Moody algebras *Nucl. Phys. B* **645** 403–37
- [25] Moody R V 1979 Root systems of hyperbolic type *Adv. Math.* **33** 144–60
- [26] Pati K C, Parashar D and Kaushal S 1998 Involutive automorphisms and Iwasawa decomposition of some hyperbolic Kac–Moody algebras *J. Math. Phys.* **40** 501–10
- [27] Peterson D H and Kac V G 1983 Infinite flag varieties and conjugacy theorems *Proc. Natl Acad. Sc. USA* **80** 1778–82
- [28] Rousseau G 1988 Formes réelles presque compactes des algèbres de Kac–Moody affines *Revue de l’Institut Elie Cartan, Nancy* **11** 175–205
- [29] Rousseau G 1988 Formes réelles presque déployées des algèbres de Kac–Moody affines *Harmonic Analysis (Luxembourg, 1987) (Springer Lecture Note in Math. vol 1359)* pp 252–64
- [30] Rousseau G 1989 Almost split K -forms of Kac–Moody algebras *Infinite Dimensional Lie Algebras and Groups (Marseille, 1988) (Adv. Ser. in Math. Physics vol 7)* ed V G Kac (Singapore: World Scientific) pp 70–85
- [31] Rousseau G 1990 Limmeuble jumelé d’une forme presque déployée d’une algèbre de Kac–Moody *Bull. Soc. Math. Belg.* **42** 673–94
- [32] Rousseau G 1994 On forms of Kac–Moody algebras *Proceedings Symposia in Pure Mathematics* vol 56 (part 2) pp 393–9
- [33] Saçlıolu C 1989 Dynkin diagrams for hyperbolic Kac–Moody algebras *J. Phys. A: Math. Gen.* **22** 3753
- [34] Satake I 1963 On the theory of reductive algebraic groups over a perfect field *J. Math. Soc. Japan* **15** 210–35
- [35] Satake I 1971 *Classification Theory of Semi-Simple Algebraic Groups (Lecture Note in Pure and Applied Math. vol 3)* (New York: Dekker)
- [36] Schnakenburg I and West P 2001 Kac–Moody symmetries of IIB supergravity *Phys. Lett. B* **517** 421
- [37] Tits J 1966 Classification of algebraic semi-simple groups *Algebraic Groups and Discontinuous Subgroups (Boulder, CO, 1965) (Proc. of Symposia in Pure Math. vol IX)* pp 33–62
- [38] Tits J 1966 Normalisateurs de Tores, groupes de Coxeter étendus *J. Algebra* **4** 96–116
- [39] Tripathy L K and Pati K C 2006 Satake diagrams of affine Kac–Moody algebras *J. Phys. A: Math. Gen.* **39** 1385–95